JEE ADVANCED

Class XII

Chapter 1

Matrices and Determinants

MATRICES

- Definition of matrix, square matrix, column and row matrix
- Algebra of matrices
- Properties
- Transpose of a matrix and its properties
- Determinant and Adjoint of matrix
- Inverse and its properties
- Symmetric and skew symmetric matrices
- Solved Examples

Matrix

- A matrix is a rectangular array of elements (usually called scalars), which are set out in rows and columns. For example, if A denotes a matrix given by $\begin{bmatrix} 2 & 5 & 7 & 8 \\ 5 & 6 & 8 & 9 \\ 2 & 5 & 6 & 4 \end{bmatrix}$, then number of rows in A are 3 and the columns are 4. And we say that it is a 3×4 matrix. More mathematically we say; an arrangement of mn numbers in m rows (horizontal lines) and n columns (vertical lines) is called an $m \times n$ matrix or a matrix of order $m \times n$. It is denoted as

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \vdots & \vdots \quad \vdots \\ a_{m1} & a_{m2} \ldots a_{mn} \end{bmatrix} \text{ or } A = [a_{ij}]_{m \times n} \text{ or } A = (a_{ij})$$

- Where a_{ij} denotes the element of i^{th} row and j^{th} column.

Type of matrices

- *Square matrix*: A matrix with equal number of rows and columns is called a square matrix. An $n \times n$ matrix is called a square matrix of order n or an n rowed matrix.
- *Row matrix*: A matrix of order $1 \times n$ is called a row matrix i.e. a row matrix have only 1 row.
- *Column matrix*: A matrix of order $m \times 1$ is called a column matrix i.e. a column matrix have only one column.
- *Zero matrix* or *null matrix* : A matrix whose all elements are zero. Matrix of such kind are denoted by $0_{m \times n}$ or 0.
- *Diagonal Matrix*: A square matrix in which all its elements are zero except those in the leading diagonal is called a diagonal matrix. Thus in a diagonal matrix $a_{ij} = 0$ if $i \neq j$
- The diagonal matrix of order 2 and 3 are as follows:

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}$$

- Note that elements a_{ij} of a matrix for which $i = j$ are called the diagonal elements of a matrix and the line along which all these elements lie is called the principal diagonal or diagonal of the matrix.
- *Scalar Matrix*: A square matrix in which all the diagonal elements are

	equal and all other elements equal to zero is called a scalar matrix i.e. in a scalar matrix $a_{ij} = k$, for $i = j$ and $a_{ij} = 0$ for $i \neq j$ • Thus $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ is a scalar matrix. • *Unit Matrix or Identity Matrix*: A square matrix in which all its diagonal elements are equal to 1 and all other elements equal to zero is called a unit matrix or identity matrix. For example $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Algebra of matrices	1. *Equality of Matrices*: Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be equal written as $A = B$ if (i) they are of same order. (ii) $a_{ij} = b_{ij}$ for each i and j. 2. *Addition of two matrices*: Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. The sum $A + B$ of A and B is defined as $A + B = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$ for each i and j. Note that we add two matrices of the same order by adding the corresponding entries. We can not add two matrices if they are of different order. 3. *Negative of a Matrix*: Let $A = [a_{ij}]_{m \times n}$ be a matrix, then the negative of the matrix A is defined as the matrix $[-a_{ij}]_{m \times n}$ and is denoted by $-A$. Thus negative of a matrix is obtained by making all the elements of the given matrix negative. 4. *Subtraction of two matrices*: Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices of the same order. Then the difference of A and B is denoted by $A - B$ $A - B = A + [-B] = [a_{ij} - b_{ij}]_{m \times n}$ Thus the matrix $A - B$ is obtained by subtracting the elements of B from the corresponding elements of A. 5. *Scalar Multiplication of a Matrix*: Let $[a_{ij}]_{m \times n}$ be a matrix and K is a scalar, then the matrix obtained by multiplying the individual element of the matrix A by K is called the scalar multiplication of A by K and is denoted by KA or AK. 5. *Multiplication of Matrices*: Let A be $m \times n$ and B be $n \times p$ matrices, then the product of matrices A and B denoted by AB is the matrix of order $m \times p$, whose $(i, j)^{th}$ element is obtained by adding the products of corresponding elements of i^{th} row of A and j^{th} column of B. Mathematically, let $A = [a_{ij}]_{m \times n}$ be a matrix of order $m \times n$ and $B = (b_{jk})_{n \times p}$ a matrix of order $n \times p$. Then the product AB is denoted as a matrix $C = [c_{ik}]_{m \times p}$ of order $m \times p$ where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ i.e. $c_{ik} = a_{i1}b_{1k} + a_{i2}\, b_{2k} + \ldots\, a_{in}b_{nk}$

Illustration 1 : Find the value of x, y, z and a which satisfy the matrix equation

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Soln.: We have given the matrix equation $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$

We have to determine the value of x, y, z and a using the given matrix equation.

From definition of equal matrices two matrices are equal if their individual elements are equal.

From the above definition of equality of matrix we will get that

$x + 3 = 0$, $2y + x = -7$, $z - 1 = 3$, $4a - 6 = 2a \Rightarrow x = -3$, $2y = -7 - x$, $\Rightarrow 2y = -7 + 3 \Rightarrow y = -2$, $z = 4$, $2a = 6 \Rightarrow a = 3 \quad \therefore\ x = -3, y = -2, z = 4, a = 3$.

Illustration 2 : If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix}; B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}; C = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 2 & 3 \\ -1 & -2 & 2 \end{bmatrix}$. Find $A - 2B + 3C$

Soln.: We have three matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix}; B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}; C = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 2 & 3 \\ -1 & -2 & 2 \end{bmatrix}$

We have to determine the value of $A - 2B + 3C$.

We will use the definition of addition and subtraction of matrix to get the required result.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \Rightarrow 2B = \begin{bmatrix} 8 & 10 & 12 \\ -2 & 0 & 2 \\ 4 & 2 & 4 \end{bmatrix} \quad \text{(By definition of scalar multiplication)}$$

$$C = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 2 & 3 \\ -1 & -2 & 2 \end{bmatrix} \Rightarrow 3C = \begin{bmatrix} -3 & -6 & 3 \\ -3 & 6 & 9 \\ -3 & -6 & 6 \end{bmatrix}$$

$$\therefore\ A - 2B + 3C = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 10 & 12 \\ -2 & 0 & 2 \\ 4 & 2 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -6 & 3 \\ -3 & 6 & 9 \\ -3 & -6 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1-8-3 & 2-10-6 & 3-12+3 \\ -1+2-3 & 0-0+6 & 2-2+9 \\ 1-4-3 & -3-2-6 & 1-4+6 \end{bmatrix} = \begin{bmatrix} -10 & -14 & -6 \\ -2 & 6 & 9 \\ -6 & -11 & 3 \end{bmatrix}$$

Illustration 3 : Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}; B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$. Find the matrix C such that $A + 2C = B$.

Soln.: $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}; B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$

We have to determine the matrix C such that $A + 2C = B$.

We will use the definition of addition and subtraction of matrix to get the required result.

$$A + 2C = B \quad \Rightarrow\ 2C = B - A \quad \Rightarrow\ C = \frac{1}{2}(B - A)$$

First we will determine $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}; A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

$$B - A = \begin{bmatrix} 3-1 & -1-2 & 2+3 \\ 4-5 & 2-0 & 5-2 \\ 2-1 & 0+1 & 3-1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\text{So } C = \frac{1}{2}(B - A) = \frac{1}{2}\begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 & 5/2 \\ -1/2 & 1 & 3/2 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

Illustration 4 : Find the value of A, B if $A - B = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$; $A + B = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$

Soln.: We have $A - B = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$; $A + B = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$

We have to determine the value of A and B.

We will solve the two equations treating them as a system of equations using definition of addition of two matrices.

$$A + B = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \quad ...(1)$$

$$A - B = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \quad ...(2)$$

Adding (1) and (2) we get $A + B + A - B = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

$$\Rightarrow \; 2A = \begin{bmatrix} 4 & 0 \\ 2 & 4 \end{bmatrix} \text{ or } A = \begin{bmatrix} 4/2 & 0 \\ 2/2 & 4/2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$$

Illustration 5 : If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, obtain the product AB and BA and show that $AB \neq BA$.

Soln.: We have two matrices $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

We have to determine the value of AB, BA and validate that $AB \neq BA$.

From definition of product of two matrices $A = [a_{ij}]_{m \times n}$ be a matrix of order $m \times n$ and $B = [b_{jk}]_{n \times p}$ be a matrix of order $n \times p$. Then the product AB is denoted as a matrix $C = [c_{ik}]_{m \times p}$ of order $m \times p$ where

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} \quad \text{i.e.} \quad c_{ik} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + ... + a_{in}\, b_{nk}.$$

We already have $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

From above definition of product of two matrix we get;

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1+0+3 & 0-2+6 & 2-4+0 \\ 2+0-1 & 0+3-2 & 4+6+0 \\ -3+0+2 & 0+1+4 & -6+2+0 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

$$\text{and } BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+0-6 & -2+0+2 & 3+0+4 \\ 0+2-6 & 0+3+2 & 0-1+4 \\ 1+4+0 & -2+6+0 & 3-2+0 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

Hence $AB \neq BA$

Illustration 6 : If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$ evaluate $A^2 - B^2$.

Soln.: We have given the matrices $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$

We have to evaluate the value of $A^2 - B^2$.

$A^2 - B^2 = (A + B)\,(A - B)$

Thus we will first determine the individual $A + B$ and $A - B$ and then multiply them to find the required value. Alternately we can determine A^2, B^2 and subtract them to get the value required.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 3 & 5 & -5 \\ 0 & -2 & 8 \end{bmatrix}_{2\times 3}$$

$$A - B = \begin{bmatrix} 1 & 1 & 7 \\ 0 & 0 & 2 \end{bmatrix}_{2\times 3}$$

Since number of columns of $A + B \neq$ number of rows of B $\therefore$ the product is not possible.

Remarks	1. The product AB is defined when the number of columns of A is equal to the number of rows of B. 2. The $(i, k)^{\text{th}}$ element c_{ik} of the product AB is given by multiplying the elements of the i^{th} row of A by the corresponding elements of the k^{th} column of B and adding the resulting products. This type of multiplication is called row by column multiplication. 3. If AB is defined, then BA may not be defined. In fact AB and BA are both defined when A and B are square matrices of the same order. In general $AB \neq BA$.

Illustration 7 : The matrix $R(t)$ is defined by $R(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$ show that $R(s)\,R(t) = R(s + t)$.

Soln.: We have the matrix $R(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

We have to show that $R(s)\,R(t) = R(s + t)$

Using the value of given $R(t)$ and the definition of product of two matrices we will find the product $R(s)$ $R(t)$ and will show that it is equivalent to $R(s + t)$.

$$R(s)\,.\,R(t) = \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = \begin{bmatrix} \cos s\cos t - \sin s\sin t & \cos s\sin t + \sin s\cos t \\ -\sin s\cos t - \cos s\sin t & -\sin s\sin t + \cos s\cos t \end{bmatrix}$$

$$= \begin{bmatrix} \cos(s + t) & \sin(s + t) \\ -\sin(s + t) & \cos(s + t) \end{bmatrix} = R(s + t)$$

Properties related to algebra of matrices	1. Matrix addition is commutative: If A and B be two $m \times n$ matrices, then $A + B = B + A$. 2. Matrix addition is Associative: If A, B and C be three $m \times n$ matrices, then $(A + B) + C = A + (B + C)$. 3. If k_1 and k_2 are scalars and A be a matrix, then $(k_1 + k_2)A = k_1A + k_2A$. 4. If k_1 and k_2 are scalars and A be a matrix, then $k_1(k_2A) = (k_1k_2)A$. 5. If A and B are two matrices of same order and k, a scalar, then $k(A + B) = kA + kB$ i.e. the scalar multiplication of matrices distributes over the addition of matrices. 6. If k_1 and k_2 are two scalars and A is any matrix, then $(k_1 + k_2)A = k_1A + k_2A$. 7. If A is any matrix and k be a scalar, then $(-k)A = -(kA) = k(-A)$. 8. (i) *Associative Law of Multiplication*: i.e. if A, B and C be three matrices of order $m \times n$, $n \times p$ and $p \times q$ respectively, then $(AB)C = A(BC)$ (ii) *Distributive Law*: Multiplication of matrices is distributive with respect to addition i.e. if A, B and C are three matrices of order $m \times n$, $n \times p$, $p \times q$ respectively, then $A(B + C) = AB + AC$. (iii) The multiplication of matrices is not always commutative.

Illustration 8 : If a, b, c, d are real numbers and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ prove that $A^2 - (a + d)A + (ad - bc)I = 0$

Soln.: We have the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for a, b, c, d being real.

We have to prove that $A^2 - (a + d)A + (ad - bc)I = 0$

We will determine A^2 and then using A^2, A and I try to solve $A^2 - (a + d)A + (ad - bc)I$ to obtain the required zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \therefore A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix}$$

$$A^2 - (a + d)A + (ad - bc)I = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix} - (a + d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad - bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc - a^2 - ad + ad - bc & ab + bd - ab - db + ad - bc \\ ac + dc - ac - dc & bc + d^2 - ad - d^2 + ad - bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Transpose of a matrix	• Let A be a $m \times n$ matrix. Then the matrix of order $n \times m$ obtained by changing its rows into columns and columns into rows is called the transpose of A and is denoted by A' or A^T or A^t. • Symbolically, if $A = [a_{ij}]_{m \times n}$ is a matrix then, the transpose of the matrix A denoted by A' or A^T or A^t is defined as $A' = [a'_{ji}]_{n \times m}$ Where $a'_{ji} = a_{ij}$ • Note that the element a_{ij} in the i^{th} row and j^{th} column of A stands in the j^{th} row and i^{th} column of A' and the transpose of $m \times n$ matrix is $n \times m$ matrix.
Some properties related to transposed matrix	(i) The transpose of the sum of two matrices is the sum of their transposes i.e. $(A + B)' = A' + B'$ (ii) The transpose of the transpose of a matrix is the matrix itself i.e. $(A^t)^t = A$ (iii) If A is any matrix and k is any number, then $(kA)' = kA'$. (iv) The transpose of the product of two matrices is the product in the reverse order of their transposes i.e. $(AB)' = B'A'$.

Illustration 9 : If $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 3 \\ 3 & -1 & 0 \end{bmatrix}$, find $A + A^T$.

Soln.: We have the matrix $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 3 \\ 3 & -1 & 0 \end{bmatrix}$

We have to find the value of $A + A^T$.

By the definition of transpose, the value of A^T is obtained by interchanging rows to columns and columns to rows.

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 3 \\ 3 & -1 & 0 \end{bmatrix} \quad \therefore \text{ By definition of transpose}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

$$\therefore \quad A + A^T = \begin{bmatrix} 2 & 6 & 5 \\ 6 & 10 & 2 \\ 5 & 2 & 0 \end{bmatrix}$$

Illustration 10 : For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$ Show that $(AB)^t = B^t \,.\, A^t$.

Soln.: We have the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$

We have to show that $(AB)^t = B^t \,.\, A^t$.

We will first determine AB and then find its transpose and show that it is equal to the product of two individual transpose.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix};\ B = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 16 & 23 \end{bmatrix} \quad \therefore (AB)^t = \begin{bmatrix} 6 & 16 \\ 9 & 23 \end{bmatrix}. \text{ Also } A^t = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}; B^t = \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}$$

$$\therefore \quad B^t A^t = \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 16 \\ 9 & 23 \end{bmatrix}$$

Determinant of a square matrix

- Corresponding to each square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots a_{2n} \\ \vdots & \vdots & \vdots \quad \vdots \\ a_{n1} & a_{n2} & a_{n3} \ldots a_{nn} \end{bmatrix}$$

there is an expression associated called the determinant of A, denoted by det A or $|A|$, written as

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \vdots & \vdots \quad \vdots \\ a_{n1} & a_{n2} \ldots a_{nn} \end{vmatrix}$$

- A matrix is an arrangement of numbers and so it has no fixed value, while each determinant has fixed value.
- A determinant having n rows and n columns is known as a determinant of order n. The determinants of non-square matrices are not defined.

Adjoint of a Matrix

- Before defining the adjoint of a matrix let us quickly discuss the ‘cofactors’ which is also given in “Determinants”.
- **Cofactors:** If A is a square matrix, then cofactor of a_{ij} in $|A|$ denoted by c_{ij} is $(-1)^{i+j}$ times the determinant obtained by leaving the i^{th} row and j^{th} column passing through the element a_{ij} in $|A|$.
- Now let us define the ‘adjoint’. Let $A = [a_{ij}]_{m \times n}$ be a square matrix and c_{ij} the cofactor of a_{ij} in $|A|$. Then the transpose of the matrix obtained by replacing each element of A by its cofactor in $|A|$ is known as adjoint of A and is denoted by Adj A.
  i.e. Adj $A = B^t$, where $B \rightarrow$ cofactor matrix $= [c_{ij}]$
- If A be any $n \times n$ square matrix then
  $A \,.\, (\text{Adj } A) = (\text{Adj } A) \,.\, A = |A|\ I_n$ where I_n is an $n \times n$ identity (unit matrix).

Illustration 11 : Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$ and verify that $A(\text{Adj}\,A) = |A|\,I_3 = (\text{Adj}\,A)\,.\,A$

Soln.: We have the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

We have to determine the Adj (A) and also have to verify that $A(\text{Adj}\,A) = |A|\,I_3 = (\text{Adj}\,A)\,A$

We know that adj (A) is the transpose of the cofactor matrix.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} \quad \therefore \; C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix} = 2$$

$$C_{12} = -\begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix} = 21 \qquad C_{13} = \begin{vmatrix} 3 & 4 \\ 0 & -6 \end{vmatrix} = -18$$

$$C_{21} = -\begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix} = 6 \qquad C_{22} = \begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix} = -7$$

$$C_{23} = -\begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix} = 6 \qquad C_{31} = \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} = 4$$

$$C_{32} = -\begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} = -8 \qquad C_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4$$

$$\therefore \quad \text{Adj}\,A = \begin{vmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{vmatrix} = \begin{vmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{vmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = 20$$

$$\text{Now } A(\text{Adj}\,A) = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} \begin{vmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{vmatrix} = \begin{vmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{vmatrix} = 20I_3 = |A|\,I_3$$

Similarly we can obtain $(\text{Adj}\,A)\,A = I_3$

Inverse of a matrix	• Let A be an $n \times n$ matrix, then an $n \times n$ matrix B is said to be inverse of A if $AB = BA = I$ (where I is $n \times n$ unit matrix). • B is also denoted as A^{-1}. So $A\,.\,A^{-1} = A^{-1}\,.\,A = I$ • Any matrix having inverse is called a invertible matrix.
Properties	(i) The inverse of a matrix if it exists is always unique. (ii) A square matrix A is invertible if $|A| \neq 0$ (iii) $A^{-1} = \dfrac{\text{adj } A}{|A|}$ (iv) A matrix A is called singular or non-singular according $|A| = 0$ or $|A| \neq 0$ (v) If A and B be two $n \times n$ invertible matrix then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$ i.e. the inverse of the product of two matrices is the product of their inverse taken in the reverse order.

Illustration 12 : Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Soln.: Given matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

We have to determine A^{-1}

$A^{-1} = \dfrac{1}{|A|}$ (Adj A), where (Adj A), is the transpose of cofactor matrix.

For given A

$$C_{11} = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = 3 \qquad C_{12} = -\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$C_{13} = \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2 \qquad C_{21} = -\begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = 2$$

$$C_{22} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 \qquad C_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = 2$$

$$C_{31} = \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = 6 \qquad C_{32} = -\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} = 2$$

$$C_{33} = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 5$$

$$\therefore \quad \text{Adj}\,A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \text{ also } |A| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} = 3 + 2 - 4 = 1$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{Adj}\,A) = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Solution of simultaneous linear equations

- There are two types of system of Linear Equations

1. Non homogeneous system of Linear equations.
2. Homogeneous system of Linear equations.

Non-homogeneous system of linear equations

- There are two basic methods for solving non-homogeneous system of linear equations namely,

1. By **Cramer's rule**: we have discussed it in our 'Determinants'. Section (page no. 23).
2. By **Matrix Method**: For the system of equations
   $AX = B$ the unknowns can be obtain as
   $X = A^{-1}B$

(i) Now if $|A| \neq 0$ then the system is consistent and has a unique solution given by $X = A^{-1}B$.

(ii) If $|A| = 0$ the system of equations has either no solution or an infinite number of solutions.
When $|A| = 0$ then find (Adj A) . B.
If (a) (Adj A) $B \neq O$, the system has no solution i.e. system is inconsistent.
If (b) (Adj A) $B = O$, the system is consistent and has infinitely many solutions.

Homogeneous system of linear equations

- System of linear equations $AX = B$ is said to be homogeneous, if $B = O$ otherwise it is known as non-homogeneous system.

1. If $|A| \neq 0$, the system of equations have only trivial solution and it has one solution.
2. If $|A| = 0$, the system of equations has non-trivial solution and it has infinite sloutions.
3. If number of equations < number of unknowns, then it has non-trivial solution.

Illustration 13 : Use matrix method to solve the following system of equations.

$$x + y + z = 3$$
$$2x - y + z = 2$$
$$x - 2y + 3z = 2$$

Soln.: Given the system of equations

$$x + y + z = 3$$
$$2x - y + z = 2$$
$$x - 2y + 3z = 2$$

We have to solve the above system to determine the value of x, y, z using matrix method.

For the system $AX = B$, if $|A| \neq 0$ then $X = A^{-1} B$

Firstly we will write the system in the form of matrix equation $AX = B$.

Thus, $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & -2 & 3 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 1(-3 + 2) - 1(6 - 1) + 1(-4 + 1) = -1 - 5 - 3 = -9 \neq 0$

$\therefore$ the equations are consistent and have a unique solution.

Now to find the solution $X = A^{-1} B$ we will determine A^{-1}.

$\text{Adj } A = \begin{bmatrix} -1 & -5 & 2 \\ -5 & 2 & 1 \\ -3 & 3 & -3 \end{bmatrix}$ (verify it)

$\therefore \quad A^{-1} = \frac{1}{|A|} \text{Adj }(A) = -\frac{1}{9} \begin{bmatrix} -1 & -5 & 2 \\ -5 & 2 & 1 \\ -3 & 3 & -3 \end{bmatrix}$

Thus $X = \begin{bmatrix} 1/9 & 5/9 & -2/9 \\ 5/9 & -2/9 & -1/9 \\ 3/9 & -3/9 & 3/9 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \therefore x = 1, y = 1, z = 1$

Illustration 14 : Use matrix method to determine whether the following systems of equations are consistent or not $4x - 3y = 5$,

$$8x - 6y = 9$$

Soln.: Given the system of equations

$$4x - 3y = 5$$
$$8x - 6y = 9$$

Using matrix method we have to determine whether the given system is consistent or not.

For the system of equation $AX = B$, the unique solution will exist if and only if $|A| \neq 0$. If $|A| = 0$ and also $(\text{Adj }(A)) B \neq 0$ then system has no solution.

The given system can be rewritten as $\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$ $\quad (AX = B)$

Now $|A| = \begin{vmatrix} 4 & -3 \\ 8 & -6 \end{vmatrix} = -24 + 24 = 0$

Thus either the system has no solution or has infinite number of solution depending on the value of $[\text{Adj}(A)]B$.

$$\therefore \quad \text{Adj}(A) = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ -8 & 4 \end{bmatrix} \quad \therefore \ (\text{Adj}\,A)\,B = \begin{bmatrix} -6 & 3 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix} \neq 0$$

$\therefore$ The system has no solution.

Symmetric and skew-symmetric matrix	• A square matrix $A = (a_{ij})$ is called a *symmetric matrix* if $A^t = A$ or $a_{ij} = a_{ji}$ for all (i, j). • A square matrix $A = (a_{ij})$ is called a skew symmetric matrix if $A^t = -A$ or $a_{ij} = -a_{ji}\ \forall\ i, j$ if $i = j$, $a_{ij} = -a_{ij} \Rightarrow -a_{ii} = a_{ii} \Rightarrow 2a_{ii} = 0$ $\Rightarrow a_{ii} = 0$ $\therefore$ the diagonal elements of a skew symmetric matrix are all zero.

Illustration 15 : If A is a square matrix, then show that

(i) $A + A^t$ is symmetric

(ii) $A - A^t$ is skew symmetric

(iii) AA^t and $A^t A$ are both symmetric matrix.

Soln.: Given that A is any square matrix.

We have to show that $A + A^t$ is symmetric, $A - A^t$ is skew symmetric and AA^t, A^tA are both symmetric matrix.

Any matrix is symmetric if $A^t = A$ and skew symmetric if $A = -A^t$.

(i) let us consider $A + A^t$.

$(A + A^t)^t = A^t + (A^t)^t = A^t + A$ (By properties of transpose of a matrix)

$\Rightarrow \ (A + A^t)^t = A^t + A$

Thus by definition of symmetric matrix $A^t + A$ is symmetric.

(ii) $(A - A^t)^t = A^t - (A^t)^t = A^t - A = -(A - A^t)$ (By properties of transpose of matrix)

$\therefore \ (A - A^t)^t = -(A - A^t)$

$\therefore A - A^t$ is skew symmetric matrix (By definition of skew symmetric matrix)

(iii) $(AA^t)^t = (A^t)^t A^t$ (using $(AB)^t = B^tA^t$)

$= AA^t$

$\therefore \ AA^t$ is symmetric matrix.

Similarly, $(A^t A)^t = A^t \cdot (A^t)^t = A^t \cdot A$

$\therefore \ A^tA$ is also symmetric matrix.

Illustration 16 : Show that the matrix $B^t AB$ is symmetric or skew symmetric iff A is symmetric or skew symmetric.

Soln.: Given the matrix $B^t AB$.

We have to show that $B^t AB$ is symmetric if A is symmetric and is skew symmetric if A is skew symmetric.

For a given square matrix A, it is symmetric if $A^t = A$ and skew symmetric if $A^t = -A$.

Let us consider

$(B^t AB)^t = [B^t (AB)]^t = (AB)^t (B^t)^t$ $(\because (AB)^t = B^t A^t)$

$= B^t A^t B = B^t A^t B$

$\therefore \ (B^t AB)^t = B^t A^t B$ ...(1)

(i) Now if A is symmetric, then $A^t = A$

$\Rightarrow$ From (1) we get

$(B^t AB)^t = B^t AB$

$\therefore B^t AB$ by definition is symmetric.

(ii) If A is skew symmetric, then $A^t = -A$

$\Rightarrow$ From (1) we get $(B^t AB)^t = B^t (-A) B = -B^t AB$

$\therefore B^t AB$ is skew symmetric.

Illustration 17 : If A and B are symmetric matrices of the same size, then

(a) $AB + BA$ is symmetric (b) $AB - BA$ is skew-symmetric.

Soln.:

We have two symmetric matrices A and B of same size (order).

We have to show that

(i) $AB + BA$ is symmetric and (ii) $AB - BA$ is skew-symmetric.

Any matrix A is symmetric if $A^t = A$ and is skew symmetric if $A^t = -A$

Let us take both of them individually. (i) $(AB + BA)^t = (AB)^t + (BA)^t = B^t A^t + A^t B^t$

Now A, B are given to be symmetric

$\therefore \; A^t = A, B^t = B \;\; \therefore \; (AB + BA)^t = BA + AB = AB + BA$

$\therefore \; (AB + BA)^t = AB + BA \quad \therefore$ By definition $AB + BA$ is symmetric.

(ii) $(AB - BA)^t = (AB)^t - (BA)^t = B^t A^t - A^t B^t$ (using $(AB)^t = B^t A^t$)

$= BA - AB$ [$\because B^t = B, A^t = A$ (given)]

$= -[AB - BA]$

$\therefore \; (AB - BA)^t = -(AB - BA) \quad \therefore AB - BA$ is skew symmetric (By definition)

Illustration 18 : Prove that inverse of a symmetric matrix if exist, is also symmetric.

Soln.: We have a matrix A such that $|A| \neq 0$ i.e. A^{-1} exist, also A is symmetric.

We have to show that A^{-1} is also symmetric.

A matrix A is symmetric if $A^t = A$ and also $A^{-1} A = A A^{-1} = I$

As $A^{-1} A = A A^{-1} = I \;\Rightarrow (A^{-1} A)^t = (A A^{-1})^t = I^t$

$\Rightarrow \; A^t . (A^{-1})^t = (A^{-1})^t . A^t = I \;\Rightarrow\; (A^t)^{-1} = (A^{-1})^t \;\Rightarrow\; (A^{-1})^t = A^{-1}$

$\therefore A^{-1}$ is symmetric by definition.

Illustration 19 : Prove that if A and B are non-singular symmetric matrices such that $AB = BA$, then (i) $A^{-1}B$, (ii) $A^{-1}B^{-1}$ are also symmetric.

Soln.: We have given two non-singular symmetric matrices A and B such that $AB = BA$.

We have to show that $A^{-1}B$ and $A^{-1}B^{-1}$ are also symmetric.

For any symmetric matrix A, if exist, A^{-1} is also symmetric, also $(AB)^{-1} = B^{-1} A^{-1}$.

Now $AB = BA$ (given)

$\Rightarrow \; (AB)^{-1} = (BA)^{-1} \Rightarrow B^{-1}A^{-1} = A^{-1}B^{-1}$ ...(1)

Now again $A^{-1}B^{-1} = B^{-1}A^{-1} \Rightarrow (A^{-1}B^{-1})\, B = (B^{-1}A^{-1})\, B$

$\Rightarrow \; A^{-1}(B^{-1}B) = B^{-1}(A^{-1}B)$

$\Rightarrow \; A^{-1} I = B^{-1}(A^{-1}B) \Rightarrow A^{-1} = B^{-1}(A^{-1}B)$

$\Rightarrow \; BA^{-1} = B[B^{-1}(A^{-1}B)] = (BB^{-1})(A^{-1}B) = I(A^{-1}B)$

$\Rightarrow \; BA^{-1} = A^{-1}B$ ...(2)

(i) $(A^{-1}B)^t = B^t(A^{-1})^t = B . A^{-1}$ ($\because B, A^{-1}$ are symmetric)

$= A^{-1}B$ (from (2))

$\therefore \; A^{-1}B$ is symmetric.

(ii) $(A^{-1}B^{-1})^t = (B^{-1})^t (A^{-1})^t = B^{-1}A^{-1} = A^{-1}B^{-1}$ (using 1) (as A^{-1}, B^{-1} are symmetric)

$\therefore A^{-1}B^{-1}$ is symmetric.

Illustration 20 : Show that every skew symmetric matrix of odd order is singular.

Soln.: Let A be a skew symmetric matrix of odd order.

We have to show that A is singular.

A matrix is singular if $|A| = 0$ and skew symmetric if $A^t = -A$.

Let A be skew symmetric matrix of order n where n is odd i.e. $n = 2m + 1$ (m is positive integer)

Since A is skew symmetric $\Rightarrow A^t = -A$

Thus $|A^t| = |-A| \;\Rightarrow\; |A| = |(-1)A|$ ($\because |A^t| = |A|$)

$\Rightarrow \; |A| = (-1)^n |A|$ (By properties of determinant)

$|A| = (-1)^{2m+1} |A| = (-1)^1 |A| = -|A|$

$\Rightarrow \; 2|A| = 0 \Rightarrow |A| = 0$

DETERMINANTS

- Definition of determinants
- Minors and co-factors
- Properties of determinants
- Multiplication of two determinants
- Differentiation of a determinant function
- Solved examples.

Definition of determinants

- Let $a_{11}, a_{12}, a_{21}, a_{22}$ are any four numbers.

  The symbol $\Delta = D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is called determinant of order 2.

  The value of D is just a single number obtained by the rule $a_{11} . a_{22} - a_{21} . a_{12}$.

  The horizontal lines are the rows of the determinant and vertical lines are the column of the determinant.

  Order: Number of rows × Number of columns.

  Thus 2 × 2 means the determinant having two rows and two columns. The determinant of order (3 × 3) has three rows and three columns.
- *Value of Determinant:* The value of determinant can be obtained by expanding along any row or column. For the determinant of order 3, the value by expanding along the first row can be obtained as follows:

  $$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

  So value is $a_{11}.\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

  This is called the expansion of the determinant along its first row. To obtain this, we multiply each element of first row by the determinant of second order which we get by leaving the row and column containing that element.
- Note that starting from the first element, the signs are positive and negative alternately.

Minors and cofactors

- **Minors:** If we delete the row and column passing through the element a_{ij} (i.e. ith row, jth column) the determinant thus obtained is called the minor of a_{ij} and denoted by M_{ij}. Consider an example of determinant of order 3.

  $$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \quad \text{Thus } M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

  Since there exist a minor corresponding to each element of the determinant, thus in total the number of minors are 9 for a determinant of order 3.
- **Cofactors:** The minor M_{ij} multiplied by $(-1)^{i+j}$ gives the cofactor of a_{ij} denoted by A_{ij}.

  Thus from above example $A_{13} = (-1)^{1+3}M_{13} = (-1)^4\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

  $$= \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
- Note that for M_{13}, 1^{st} row and 3^{rd} column are deleted.

Properties of determinants

1. The value of determinant does not change when rows and columns are interchanged. The determinant obtained by interchanging rows and columns is called transpose of the determinant (Δ' or Δ^T)

Thus, $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \Delta'$ or (Δ^T)

2. If all the elements of row (or column) are zero, then determinant is also zero.
3. Interchanging of any two rows (or columns) changes the sign of determinants, though the numerical value remains unchanged.

i.e. for $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

and we inter-change first and third row (denoted $R_1 \leftrightarrow R_3$), then

$\Delta_1 = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = -\Delta$

4. If a row (or column) is multiplied by a number k, then determinant is also multiplied by k. i.e.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and $\Delta' = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$\Rightarrow \Delta' = k\Delta$

5. If any two rows (columns) are identical or proportional to each other, then the value of determinant is zero.
6. If every element of any row (column) is sum of two numbers, the determinant can be expressed as the sum of two determinants of same order.

i.e. $\begin{vmatrix} a_1+\alpha_1 & a_2+\alpha_2 & a_3+\alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

7. The value of determinant remains unaltered under an operation of the form

$$R_i \to R_i + pR_j + qR_k;\ i \neq j \neq k$$

Similar operations can be performed on columns too. Note that more than one operation can be used simultaneously but one row or column must remain unchanged.

8. If a determinant $\Delta(x)$ becomes zero on putting $x = a$, then $(x - a)$ is a factor of $\Delta(x)$.
9. Determinant which have all its entries zero except the diagonal elements, is equal to the product of its diagonal elements.

Illustration 21 : Evaluate: $\Delta = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$, where ω is the cube root of unity.

Soln.: Given the determinant $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$, where ω is cube root of unity.

We have to determine the value of determinant.

We will try to make the maximum number of zero possible in any of the row or column to minimize the expansion. Also ω being cube root of unity must satisfy $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$

Operating $R_1 \to R_1 + R_2 + R_3$ we get

$$\begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

Thus by properties of determinant, value of the determinant is zero.

Illustration 22 : If $p + q + r = 0$, show that $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Soln.: Given that $p + q + r = 0$

We have to show that $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

We will expand one of the determinant and using the given condition $p + q + r = 0$, try to obtain the another determinant.

Let us take determinant on L.H.S. Expanding by Ist row we get;

$$pa\begin{vmatrix} ra & pb \\ pc & qa \end{vmatrix} - qb\begin{vmatrix} qc & pb \\ rb & qa \end{vmatrix} + rc\begin{vmatrix} qc & ra \\ rb & pc \end{vmatrix} = pa\,(qra^2 - p^2bc) - qb\,(q^2ac - b^2rp) + rc\,(c^2qp - r^2ba)$$

$= a^3prq - p^3abc - q^3abc + b^3rpq + rc^3pq - r^3abc = pqr\,(a^3 + b^3 + c^3) - abc\,(p^3 + q^3 + r^3)$

since $p + q + r = 0 \Rightarrow p^3 + q^3 + r^3 - 3pqr = 0 \Rightarrow p^3 + q^3 + r^3 = 3pqr$

$\therefore$ we have $pqr\,(a^3 + b^3 + c^3 - 3abc)$ ...(A)

Similarly expanding determinant on R.H.S. i.e.

$pqr\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ we get $pqr(a^3 + b^3 + c^3 - 3abc)$ ...(B)

Thus from (A) and (B) L.H.S. = R.H.S.

Illustration 23 : Evaluate: $\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

Soln.: Given the determinant $\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

We have to evaluate the value of the above given determinant.

By using the various rows and columns operation and the properties of determinants we will try to make as many entries possible to be equal to zero so as to minimize the expansion process.

Applying $C_2 \to C_2 + C_3$ we get

$$\begin{vmatrix} 1 & bc+ab+ac & a(b+c) \\ 1 & ca+bc+ab & b(c+a) \\ 1 & ab+ac+cb & c(a+b) \end{vmatrix} = (ab+ac+bc)\begin{vmatrix} 1 & 1 & a(b+c) \\ 1 & 1 & b(c+a) \\ 1 & 1 & c(a+b) \end{vmatrix}$$

Since $C_1 = C_2 \Rightarrow$ Value of determinant must be zero.

Illustration 24 : If $a + b + c = 2s$ and $A = \begin{bmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{bmatrix}$, find det A.

Soln.: If $s = 0$, $\det A = a^2b^2c^2\begin{vmatrix}1&1&1\\1&1&1\\1&1&1\end{vmatrix} = 0$

$\therefore$ s^2 is a factor of $\Delta = \det A$ since 3 rows are identical.

$$s = a \Rightarrow \Delta = \begin{vmatrix}a^2 & 0 & 0\\(a-b)^2 & b^2 & (a-b)^2\\(a-c)^2 & (a-c)^2 & c^2\end{vmatrix} = \begin{vmatrix}a^2 & 0 & 0\\c^2 & b^2 & c^2\\b^2 & b^2 & c^2\end{vmatrix} \text{ since } b + c = a = 0$$

$\therefore s - a$ is a factor of Δ. Likewise $s - b$ and $s - c$ are also factors. But Δ is a sixth degree polynomial. The sixth factor is of the form $k(a + b + c)$

$$\therefore \begin{vmatrix}a^2 & (s-a)^2 & (s-a)^2\\(s-b)^2 & b^2 & (s-b)^2\\(s-c)^2 & (s-c)^2 & c^2\end{vmatrix} = k\,(a + b + c)\, s^2\,(s - a)\,(s - b)\,(s - c).$$

To determine k, we set $a = b = 0$, $c = 2 \Rightarrow s = 1$. This identity becomes

$$\begin{vmatrix}0&1&1\\1&0&1\\1&1&4\end{vmatrix} = -2k \qquad \Rightarrow \quad -2 = -2k \Rightarrow k = 1$$

$\therefore$ $\Delta = 2s^3\,(s - a)\,(s - b)\,(s - c)$.

Illustration 25 : Without expanding show that $\begin{vmatrix}1 & a & a^2 - bc\\1 & b & b^2 - ca\\1 & c & c^2 - ab\end{vmatrix} = 0$

Soln.: Given determinant $\begin{vmatrix}1 & a & a^2 - bc\\1 & b & b^2 - ca\\1 & c & c^2 - ab\end{vmatrix}$

Our Aim is to show that the value of above given determinant is zero.

Since we have to find the value of given determinant without expanding it, thus the procedure of solving this problem will involve using various properties of determinants. These properties have to be used in such a manner so that they make maximum zero in the determinant.

$$\text{L.H.S.} = \underset{D_1}{\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix}} + \underset{D_2}{\begin{vmatrix}1 & a & -bc\\1 & b & -ac\\1 & c & -ab\end{vmatrix}}$$

Consider D_1 first and apply $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$.

$$D_1 = \begin{vmatrix}1 & a & a^2\\0 & a-b & a^2-b^2\\0 & b-c & b^2-c^2\end{vmatrix} = (a-b)(b-c)\begin{vmatrix}1 & a & a^2\\0 & 1 & a+b\\0 & 1 & b+c\end{vmatrix} \quad (\text{Now apply } R_2 \to R_2 - R_3)$$

$$= (a-b)(b-c)\begin{vmatrix}1 & a & a^2\\0 & 0 & a-c\\0 & 1 & b+c\end{vmatrix} = (a-b)(b-c)(c-a)$$

Similarly $D_2 = -(a-b)(b-c)(c-a)$. Thus $D_1 + D_2 = 0$.

Illustration 26 : If $a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$, $c = a\cos B + b\cos A$, find

(*i*) $b^2 + c^2 - a^2$

(*ii*) $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C$

(*iii*) $a^2 : b^2 : c^2$

Soln.: (*i*) $a = b\cos C + c\cos B$ ... (1)

$b = c\cos A + a\cos C$ ... (2)

$c = a\cos B + b\cos A$ ... (3)

$b\,(2) + c\,(3) - a\,(1) \Rightarrow b^2 + c^2 - a^2 = 2bc\cos A.$

(*ii*) The system of equation (1) to (3) $\Rightarrow \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} = 0$

$\Rightarrow \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$ ...(4)

(*iii*) (1) and (2)

$\Rightarrow -a + b\cos C + c\cos B = 0$

$a\cos C - b + c\cos A = 0$

$$\Rightarrow \frac{a}{\cos A\cos C + \cos B} = \frac{b}{\cos B\cos C + \cos A} = \frac{c}{\sin^2 C}$$

$\therefore a^2 : b^2 : c^2 = (\cos A\cos C + \cos B)^2 : (\cos B\cos C + \cos A)^2 : \sin^4 C$

$= (1 - \cos^2 A)(1 - \cos^2 C) : (1 - \cos^2 B)(1 - \cos^2 C) : \sin^4 C$ using (4)

$\therefore a^2 : b^2 : c^2 = \sin^2 A : \sin^2 B : \sin^2 C.$

Illustration 27 : If a, b, c are in G.P. or $x - \alpha$ is a factor of $ax^2 + 2bx + c = 0$, then show that

$$\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0$$

Soln.: Given that a, b, c are in G.P. and $x - \alpha$ is a factor of $ax^2 + 2bx + c = 0$

We have to show that $\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0$

Since a, b, c are in G.P. $\Rightarrow ac = b^2$

also $x - \alpha$ is a factor of $ax^2 + 2bx + c = 0 \Rightarrow a\alpha^2 + 2b\alpha + c = 0$

Let $\Delta = \begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix}$

Now by applying $R_3 \to R_3 - (\alpha R_1 + R_2)$

$\therefore$ Determinant will become $\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ 0 & 0 & -(a\alpha^2 + 2b\alpha + c) \end{vmatrix}$

$\therefore$ Expanding along third row we get $-(a\alpha^2 + 2b\alpha + c)(ac - b^2)$

Now since $ac = b^2$ [$\because a, b, c$ are in G.P.]

or $a\alpha^2 + 2b\alpha + c = 0$ [$\because x - \alpha$ is factor of $ax^2 + 2bx + c = 0$]

$\therefore \Delta = 0.$

Illustration 28 : If $\begin{vmatrix} \frac{1}{a+x} & \frac{1}{b+x} & \frac{1}{c+x} \\ \frac{1}{a+y} & \frac{1}{b+y} & \frac{1}{c+y} \\ \frac{1}{a+z} & \frac{1}{b+z} & \frac{1}{c+z} \end{vmatrix} = \frac{P}{Q}$

where Q is the product of the denominators, find P.

Soln.: $P = \begin{vmatrix} x^2+x(b+c)+bc & x^2+x(c+a)+ca & x^2+x(a+b)+ab \\ y^2+y(b+c)+bc & y^2+y(c+a)+ca & y^2+y(a+b)+ab \\ z^2+z(b+c)+bc & z^2+z(c+a)+ca & z^2+(a+b)z+ab \end{vmatrix}$ $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$= (y - x)(z - x)\begin{vmatrix} x^2+x(b+c)+bc & x^2+x(c+a)+ca & x^2+x(a+b)+ab \\ x+y+b+c & x+y+c+a & x+y+a+b \\ x+z+b+c & x+z+c+a & x+z+a+b \end{vmatrix}$ $C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$

$= (y - x)\ (z - x)\ (a - b)\ (a - c)\begin{vmatrix} x^2+x(b+c)+bc & x+c & x+b \\ x+y+b+c & 1 & 1 \\ x+z+b+c & 1 & 1 \end{vmatrix}$ $C_3 \to C_3 - C_2$

$= (y - x)(z - x)(a - b)(a - c)\begin{vmatrix} x^2+x(b+c)+bc & x+c & x+b \\ x+y+b+c & 1 & 1 \\ z-y & 0 & 0 \end{vmatrix}$

$= (y - x)\ (z - x)\ (a - b)\ (a - c)\ (z - y)\ (c - b)$

$= (x - y)(y - z)\ (z - x)\ (a - b)\ (b - c)\ (c - a).$

Illustration 29 : Prove that: $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$

Soln.: Given: $\Delta_1 = \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$

We have to show that the value of above given determinant is $4xyz$.

Procedure involving the solution of the given determinant using elementary row and column operations.

By applying: $C_1 \to C_1 - (C_2 + C_3)$ we get,

$\begin{vmatrix} 0 & z & y \\ -2x & z+x & x \\ -2x & x & x+y \end{vmatrix}$

Now by applying $R_2 \to R_2 - R_3$

$\begin{vmatrix} 0 & z & y \\ 0 & z & -y \\ -2x & x & x+y \end{vmatrix}$. Now expanding along C_1, we get; $\Delta_1 = -2x(-yz - yz) = 4xyz.$

Illustration 30 : Prove that: $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left[1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right]$

Hence find the value of determinant if a, b, c, d are the roots of the equation $px^4 + qx^3 + rx^2 + sx + t = 0$

Soln.: Given the determinant $\Delta = \begin{vmatrix} a+1 & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$

We have to evaluate the value of determinant which must be $abcd\left[1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right]$.

Moreover we have to find the value of determinant if it is given that a, b, c, d are the roots of the equation $px^4 + qx^3 + rx^2 + sx + t = 0$

Procedure involves solving the determinant using elementary row and column operations. Also we already know that for a, b, c, d be the roots of $px^4 + qx^3 + rx^2 + sx + t = 0$.

$\sum a = -\frac{q}{p}$; $\sum ab = \frac{r}{p}$; $\sum abc = -\frac{s}{p}$ and $abcd = \frac{t}{p}$.

Taking a common from R_1, b from R_2, c from R_3 and d from R_4 we get,

$$\Delta = abcd \begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

Now applying $R_1 \to R_1 + R_2 + R_3 + R_4$ and taking $1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$ common from R_1 we get

$$abcd\left[1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right] \begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

Now by applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - C_1$

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{b} & 1 & 0 & 0 \\ \frac{1}{c} & 0 & 1 & 0 \\ \frac{1}{d} & 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \qquad \text{...(A)}$$

$$\Rightarrow \quad \Delta = abcd + bcd + acd + abd + abc = abcd + \sum abc$$

(Since a, b, c, d are roots of $px^4 + qx^3 + rx^2 + sx + t = 0$)

$$\Rightarrow \quad \Delta = \frac{t}{p} - \frac{s}{p} = \frac{t-s}{p}$$

Multiplication of two determinants	• The multiplication of two determinants is only possible when both the determinants have same order. Consider two determinants of order 2 i.e. $A = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ $B = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}$ • Thus, $AB = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}$ • Now to get the $(i, j)^{\text{th}}$ element of the product we multiply the i^{th} row of A by j^{th} row of B. i.e. $AB = \begin{vmatrix} a_1\alpha_1 + a_2\beta_1 & a_1\alpha_2 + a_2\beta_2 \\ b_1\alpha_1 + b_2\beta_1 & b_1\alpha_2 + b_2\beta_2 \end{vmatrix}$

Differentiation of a determinant function	• If given $F(x)=\begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}$ Differentiating w.r.t. x we get $F'(x)=\begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix}+\begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}$ • Similarly for determinant of order 3, we get the sum of three determinants.

Illustration 31 : If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$, then show that $\Delta_1' = 3\Delta_2$, where Δ_1' is the first derivate of Δ_1 with respect to x.

Soln.: Two determinants $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are given

We know that $\Delta' = \Delta(R_1', R_2, R_3) + \Delta(R_1, R_2', R_3) + \Delta(R_1, R_2, R_3')$

$$\Delta_1' = \begin{vmatrix} 1 & 0 & 0 \\ a & x & b \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ 0 & 1 & 0 \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ a & x & b \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} = 3\Delta_2$$

Illustration 32 : If $f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin(2x^2) \end{vmatrix}$, find the value of $f'(0)$.

Soln.: Given the determinant $f(x)$.

We have to evaluate $f'(0)$ for given $f(x)$.

First we can differentiate the determinant w.r.t. x and then substitute $x = 0$, or we can first evaluate the determinant and then find $f'(x)$ at $x = 0$.

$$f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin(2x^2) \end{vmatrix}$$

$f'(x) = \Delta(R_1', R_2, R_3) + \Delta(R_1, R_2', R_3) + \Delta(R_1, R_2, R_3')$

Now we will evaluate these three determinants one by one at $x = 0$.

$$\Delta(R_1', R_2, R_3) = \begin{vmatrix} -(1+2x)\sin(x+x^2) & (1+2x)\cos(x+x^2) & (1+2x)\sin(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$$

$\therefore\ \Delta(R_1', R_2, R_3)$ at $x = 0$ is $\begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$

Similarly $\Delta(R_1, R_2', R_3) = 0$ at $x = 0$

Now $\Delta(R_1, R_2, R_3') = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ 2\cos 2x & 0 & 4x\cos 2x^2 \end{vmatrix}$ at $x = 0$, $\Delta(R_1, R_2, R_3') = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 2$

$\therefore f'(x) = 2.$

Illustration 33 : If $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$, prove that $\int_0^{\pi/2} f(x)\,dx = \frac{15\pi + 32}{60}$.

Soln.: Given the determinant $f(x)$

We have to prove that $\int_0^{\pi/2} f(x)\,dx = \frac{15\pi + 32}{60}$ for given $f(x)$.

Procedure involves first evaluating the determinant, followed by the integration.

By applying $R_1 \to R_1 - R_3 \sec x$ and $R_2 \to R_2 - R_3 \cos^2 x$, we get

$$f(x) = \begin{vmatrix} 0 & 0 & \sec^2 x + \frac{\cos x}{\sin^2 x} - \cos x \\ 0 & \cos^2 x - \cos^4 x & \operatorname{cosec}^2 x - \cos^4 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Now expanding along C_1 we get; $f(x) = -\cos^2 x \sin^2 x \left[\frac{1}{\cos^2 x} + \frac{\cos x}{\sin^2 x} - \cos x\right]$

$= -[\sin^2 x + \cos^3 x - \cos^3 x \sin^2 x] = -[\sin^2 x + \cos^3 x - \cos^3 x + \cos^5 x] = -\sin^2 x - \cos^5 x$

$\therefore \int_0^{\pi/2} f(x)\,dx = -\int_0^{\pi/2} \sin^2 x\,dx - \int_0^{\pi/2} \cos^5 x\,dx$ (Gamma integral)

$= -\frac{\pi}{4} - \frac{8}{15}$

Illustration 34 : If $y = \sin px$ and $f(x) = \begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$ where $y_n = \frac{d^n y}{dx^n}$, then find $f'(x)$.

Soln.: Given $y = \sin px$ and $f(x) = \begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$ where $y_n = \frac{d^n y}{dx^n}$

We have to find $f'(x)$.

First we will find $f(x)$ by substituting various y_i's and then find $f'(x)$.

$$y = \sin px \Rightarrow f(x) = \begin{vmatrix} \sin px & p\cos px & -p^2 \sin px \\ -p^3 \cos px & p^4 \sin px & p^5 \cos px \\ -p^6 \sin px & -p^7 \cos px & p^8 \sin px \end{vmatrix} = p^9 \begin{vmatrix} \sin px & p\cos px & -p^2 \sin px \\ -\cos px & p\sin px & p^2 \cos px \\ -\sin px & -p\cos px & p^2 \sin px \end{vmatrix}$$

Now by applying $R_1 \to R_1 + R_3$

We get $f(x) = 0 \Rightarrow f'(x) = 0$

Quick revision

1. If all the elements of the determinant are real numbers, it is called real determinant and if at least one element is imaginary ($a + i\,b$, $b \neq 0$) it is called imaginary determinant.
2. We can find the value of determinant by expanding along any row or column.
3. Minors are an order less than the order of the determinant.
4. The determinant remains unaltered by interchanging rows and columns.
5. Value of determinant having all its entries zero, is zero.
6. If all elements of a row (column) are equal or proportional to other row (column), then determinant is zero.
7. The interchanging of any two rows (columns) results in the change of the sign of the determinant.

	8. All the properties of determinants are equally applicable to both rows and columns but independently. 9. All the rows (columns) sshould not be disturbed at a time. 10. It is always desirable to try to bring in as many zeros in the rows (columns) to make the expansion along that row (column) easy.
Theory of linear equations	• Definition **homogeneous** and **non-homogeneous system** • **Cramer's rule** • Condition for **consistent (unique)** solution, **inconsistency** (no solution) and **infinitely many solutions**.
System of linear equations	Let us consider three equations $ax + by + cz = d$ $a_1x + b_1y + c_1z = d_1$ $a_2x + b_2y + c_2z = d_2$ is called a system of linear equations in three unknowns x, y, z. • The system is homogeneous if $d_1 = d_2 = d = 0$ whereas, if there exist at least one $d \neq 0$, then the above system is known as non-homogeneous.
Cramer's rule	• Consider the system of linear equations $a_1x + b_1y + c_1z = d_1$ $a_2x + b_2y + c_2z = d_2$ $a_3x + b_3y + c_3z = d_3$ $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Δ_x is obtained by supressing the column of coefficient of x by d_1, d_2, d_3. i.e. $\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ Similarly, $\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$; $\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$ Then for $x = \frac{\Delta_x}{\Delta},\ y = \frac{\Delta_y}{\Delta},\ z = \frac{\Delta_z}{\Delta}$ This is known as Cramer's rule.

Illustration 35 : If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2, and let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$.

Prove that $\Delta(x)$ is a polynomial of degree at most 2.

Soln.: Given three polynomials $f(x)$, $g(x)$, $h(x)$ of degree 2 and $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$

We have to show that with the above conditions $\Delta(x)$ is a polynomial of degree at most 2.

We will first take $f(x)$, $g(x)$, $h(x)$ to be some arbitrary polynomials of degree 2 and find $\Delta(x)$.

Let $f(x) = ax^2 + bx + c,\ g(x) = Ax^2 + Bx + C,\ h(x) = mx^2 + nx + q$

$$\therefore\ \Delta(x) = \begin{vmatrix} ax^2+bx+c & Ax^2+Bx+C & mx^2+nx+q \\ 2ax+b & 2Ax+B & 2mx+n \\ 2a & 2A & 2m \end{vmatrix}. \text{ Applying } R_2 \to R_2 - xR_3$$

$$\Delta(x) = 2\begin{vmatrix} f(x) & g(x) & h(x) \\ b & B & n \\ a & A & m \end{vmatrix}$$

$\Rightarrow \Delta(x) = 2\,[(Bm - nA)f(x) - g(x)(bm - an) + h(x)(bA - aB)]\quad \therefore$ degree $\Delta(x) \le 2$

Conditions for consistent, inconsistent and infinite solutions for homogeneous equations	• Consider the homogeneous linear equations in three unknowns x, y, z $a_1x + b_1y + c_1z = 0$ $a_2x + b_2y + c_2z = 0$ $a_3x + b_3y + c_3z = 0$ Now (i) if $\Delta \neq 0$ i.e. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then there is only one solution i.e. $x = y = z = 0$, which is also known as trivial solution. (ii) If $\Delta = 0$, the system has infinite solutions and so a non-zero solution exist. To find the solution we can use the elimination method i.e. assign some arbitrary value to one of the variable and find the value of other two in terms of that to obtain the infinitely many solutions depending on the value of that arbitrary constant.
Non homogeneous linear equations	From Cramers rule $x = \frac{\Delta_x}{\Delta};\ y = \frac{\Delta_y}{\Delta};\ z = \frac{\Delta_z}{\Delta}$ ...(A) (i) If $\Delta \neq 0$ unique solution given by (A) exist. (ii) If $\Delta = 0$ and $\Delta_x, \Delta_y, \Delta_z$ are all zero, then the system of equations has infinitely many solutions that can be obtained by elimination method. (iii) If $\Delta = 0$ and $\Delta_x, \Delta_y, \Delta_z$ are not all zero, then system must be inconsistent i.e. have no solution.

Illustration 36 : If a, b, c are all different, solve the system of equations: $x + y + z = 1$, $ax + by + cz = k$, $a^2x + b^2y + c^2z = k^2$.

Soln.: Given the system of equations $x + y + z = 1$

$ax + by + cz = k$, where $a \neq b \neq c$

$a^2x + b^2y + c^2z = k^2$

We have to solve the above system of equations.

for $\Delta \neq 0$

$$x = \frac{\Delta_x}{\Delta};\ y = \frac{\Delta_y}{\Delta};\ z = \frac{\Delta_z}{\Delta}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix} = (b-a)(c-a)(c-b)$$

$\therefore$ Since $a \neq b \neq c \Rightarrow \Delta \neq 0\ \therefore$ By Cramer's rule

$$\Delta_x = \begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix} = (k-b)(b-c)(c-k)$$

$$\Delta_y = \begin{vmatrix} 1 & 1 & 1 \\ a & k & c \\ a^2 & k^2 & c^2 \end{vmatrix} = (a-k)(k-c)(c-a) \text{ and } \Delta_z = (a-b)(b-k)(k-a)$$

$$\therefore\ x = \frac{(k-b)(c-k)}{(a-b)(c-a)};\ y = \frac{(a-k)(k-c)}{(a-b)(b-c)} \text{ and } z = \frac{(b-k)(k-a)}{(b-c)(c-a)}$$

Illustration 37 : Find the value of k if the following equations are consistent: $x + y - 3z = 0$, $(1 + k)x + (2 + k)y - 8z = 0$, $x - (k + 1)y + (2 + k)z = 0$

Soln.: Given the system of three linear equations in three variables x, y, z

We have to determine the value of k such that the system is consistent.

The system of three linear equations in three variables is consistent only if $\Delta = 0$.

$$\Delta = \begin{vmatrix} 1 & 1 & -3 \\ 1+k & 2+k & -8 \\ 1 & -(1+k) & 2+k \end{vmatrix} = 0$$

By applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 2C_1$ we get $\begin{vmatrix} 1 & 0 & 0 \\ 1+k & 1 & 3k-5 \\ 1 & -2-k & k+5 \end{vmatrix} = 0$

Now expanding along I[st] row we get; $(k + 5) - (-2 - k)(3k - 5) = 0$

$\Rightarrow\ 3k^2 + 2k - 5 = 0\ \Rightarrow\ (k - 1)(3k + 5) = 0\ \Rightarrow\ k = 1, -5/3$

Illustration 38 : Given $a = \dfrac{x}{y-z}$, $b = \dfrac{y}{z-x}$, $c = \dfrac{z}{x-y}$ where x, y, z are all non-zero and distinct. Prove that $1 + ab + bc + ac = 0$.

Soln.: Given the system of equation $a = \dfrac{x}{y-z};\ b = \dfrac{y}{z-x};\ c = \dfrac{z}{x-y}$

where $x \neq 0; y \neq 0; z \neq 0$ and $x \neq y \neq z$ $\therefore$ the system has non-zero solution.

We have to prove with the above given conditions that $1 + ab + bc + ca = 0$.

For the homogeneous system of equations, the non zero solution will exist only if $\Delta = 0$

$a = \dfrac{x}{y-z} \Rightarrow x - ay + az = 0$ ...(i)

$b = \dfrac{y}{z-x} \Rightarrow bx + y - bz = 0$ ...(ii)

$c = \dfrac{z}{x-y} \Rightarrow cx - cy - z = 0$ ...(iii)

The equations (i), (ii) and (iii) are consistent in x, y, z only if $\Delta = 0$

i.e. $\begin{vmatrix} 1 & -a & a \\ b & 1 & -b \\ c & -c & -1 \end{vmatrix} = 0 \Rightarrow 1 + ab + bc + ca = 0.$

Illustration 39 : Let $A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$. Find the non-zero column vectors X such that $AX = \lambda X$ for some scalar λ.

Soln.: $\begin{vmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda - 1)(\lambda - 2) = 0$

If $\lambda = 0$, we get $11x - 4y - 7z = 0$

$7x - 2y - 5z = 0$, $10x - 4y - 6z = 0$ giving, $\dfrac{x}{1} = \dfrac{y}{1} = \dfrac{y}{1}$, $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

If $\lambda = 1$, we get $10x - 4y - 7z = 0$

$7x - 3y - 5z = 0$, $10x - 4y - 7z = 0$ giving, $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$, $X = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

If $\lambda = 2$, we get $9x - 4y - 7z = 0$

$7x - 4y - 5z = 0$

$10x - 4y - 8z = 0$ giving, $\frac{x}{2} = \frac{y}{1} = \frac{z}{2}$, $X = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

The vectors are $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Illustration 40 : For what value of p and q, the system of equations $2x + py + 6z = 6$, $x + 2y + qz = 5$, $x + y + 3z = 4$ has (i) no solution (ii) a unique solution (iii) Infinitely many solutions

Soln.: Given the system of equations

$$2x + py + 6z = 6$$
$$x + 2y + qz = 5$$
$$x + y + 3z = 4$$

We have to determine the value of p and q such that

(i) there exist no solution

(ii) A unique solution

(iii) Infinitely many solutions

The non homogeneous system of linear equation has a

(i) unique solution if $\Delta \neq 0$

(ii) No solution if $\Delta = 0$ and any of Δ_x, Δ_y, $\Delta_z \neq 0$

(iii) Infinitely many solutions if $\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$

The given system of equation is

$$2x + py + 6z = 8$$
$$x + 2y + qz = 5$$
$$x + y + 3z = 4$$

$$\Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = (2 - p)(3 - q)$$

$$\Delta_x = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix} = 30 - 8q - 15p + 4pq$$

$$\Delta_y = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix} = 0$$

$$\Delta_z = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = p - 2$$

Now (i) for unique solution $\Delta \neq 0 \Rightarrow p \neq 2; q \neq 3$

(ii) for no solution $\Delta = 0$; $\Delta_x \neq 0$ or $\Delta_z \neq 0 \Rightarrow q = 3, p \neq 2$

(iii) for infinitely solutions $\Delta = 0$, $\Delta_x = \Delta_y = \Delta_z = 0 \Rightarrow p = 2;\ q \in R$

Competitions Window

- **Upper Triangular Matrix**
  A square matrix $A = [a_{ij}]$ is called an upper triangular matrix, if $a = 0$ for all $i > j$.
- **Lower Triangular Matrix**
  A square matrix $A = [a_{ij}]$ is called a lower triangular matrix, if $a_{ij} = 0$ for all $i < j$.
- **Trace of a Matrix**
  The sum of the diagonal elements of a square matrix A is called the trace of A and is denoted by $tr(A)$.
- **Submatrix**
  A matrix which is obtained from a given matrix by deleting any number or rows or columns or both is called a submatrix of the given matrix.
- **Equality of Matrices**
  Two matrices A and B are said to be equal, if they are of same order and all the corresponding elements are equal. It is written as $A = B$.
- **Transpose of a Matrix**
  Let A be a $m \times n$ matrix. Then, the $n \times m$ matrix obtained by interchanging the rows and columns of A is called the transpose of A and is denoted by A' or A^t or A^c. Thus
  (i) If order of A is $m \times n$, then order of A' will be $n \times m$.
  (ii) $(i, j)^{\text{th}}$ element of $A = (j, i)^{\text{th}}$ element of A'.
- **Properties of the Transpose of a Matrix**
  (i) $(A')' = A$
  (ii) $(A + B)' = A' + B'$
  (iii) $(AB)' = B'A'$
  (iv) $(kA)' = kA'$
  (v) $(A^n)' = (A')^n$
  (vi) Trace A' = Trace A
  (vii) Trace $AA' \geq 0$
- **Symmetric matrix**
  A square matrix A is said to be symmetric if, $A' = A$. That is, the matrix $A = [a_{ij}]_{n \times n}$ is said to be symmetric provided $a_{ij} = a_{ji}$ for all i and j.
- **Skew-Symmetric Matrix**
  A square matrix A is said to be skew-symmetric, if $A' = -A$. That is, the matrix $A = [a_{ij}]_{m \times n}$ is skew-symmetric if $a_{ij} = -a_{ji}$ for all i and j.
- **Properties of Symmetric and Skew-Symmetric Matrices**
  (i) If A is a square matrix, then
     (a) $A + A'$ is a symmetric matrix, and
     (b) $A - A'$ is a skew-symmetric matrix
  (ii) If A and B are two symmetric (or skew-symmetric) matrices of the same order, then so is A + B.
  (iii) If A is symmetric (or skew-symmetric) matrix and k is scalar, then kA is also symmetric (or skew-symmetric)
  (iv) If A and B are symmetric matrices of the same order, then the product AB is symmetric, if and only if $AB = BA$.

(v) Every square matrix can be expressed uniquely as the sum of a symmetric and a skew-symmetric matrix.
(vi) The matrix $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.
(vii) All positive integral powers of a symmetric matrix are symmetric.
(viii) All positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric.
(xi) If A and B are symmetric matrices of the same order, then
(a) $AB - BA$ is a skew-symmetric matrix and
(b) $AB + BA$ is a symmetric matrix.

- **Orthogonal Matrix**
A square matrix of order $n \times n$ is said to be orthogonal, if $AA' = I_n = A'A$.

- **Properties of Orthogonal Matrices**
(i) If A is an orthogonal matrix, then A' is also orthogonal.
(ii) For any two orthogonal matrices A and B, AB is an orthogonal matrix.
(iii) For any two orthogonal matrices A and B, BA is an orthogonal matrix.
(vi) if A is an orthogonal matrix, then A^{-1} is also orthogonal.

- **Singular Matrix**
A square matrix A is said to be a singular matrix, if determinant of A denoted by det A or $|A|$ is zero, *i.e.*, $|A| = 0$, otherwise, it is a non-singular matrix.

- **Idempotent Matrix**
A square matrix A is said to be idempotent if, $A^2 = A$.

- **Properties of Idempotent Matrices**
(i) If A and B are idempotent matrices, then AB is an idempotent, iff $AB = BA$.
(ii) If A and B are idempotent matrices, then $A + B$ is an idempotent, iff $AB = BA = O$.
(iii) If A is an idempotent and $A + B = I$, then B is an idempotent and $AB + BA = O$.
(vi) Diagonal (1, 1, 1, ..., 1) is an idempotent matrix.
(v) If $AB = A$ and $BA = B$, then $A^2 = A$, $B^2 = B$.
(vi) If l_1, l_2, l_3 are direction cosines, then $\begin{bmatrix} l_1^2 & l_1 l_2 & l_1 l_3 \\ l_1 l_2 & l_2^2 & l_2 l_3 \\ l_3 l_1 & l_3 l_2 & l_3^2 \end{bmatrix}$ is an idempotent as $|\Delta|^2 = 1$.

- **Involutory Matrix**
A square matrix A is said to be involutory matrix, if $A^2 = I$.

- **Nilpotent Matrix**
A square matrix A is said to be nilpotent matrix, if there exists a positive integer m such that $A^m = O$. If m is the least positive integer such that $A^m = O$, then m is called the index of the nilpotent matrix A.

- **Conjugate of a Matrix**
The matrix obtained from any given matrix A containing complex numbers as its elements, on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$.

- **Properties of Conjugate**
(i) If A be any matrix, then $\overline{(\bar{A})} = A$.
(ii) If A and B be matrices of same order, then $\overline{(A + B)} = \bar{A} + \bar{B}$.
(iii) If A be any $m \times n$ matrix and B be any $n \times p$ matrix, then $\overline{(AB)} = \bar{A} \cdot \bar{B}$.

(iv) If A be any matrix and k be any scalar, then $\overline{(kA)} = k\bar{A}$.

(v) If A be any square matrix, then $\overline{(A^n)} = (\bar{A})^n$.

- **Transpose Conjugate of a Matrix**
  The transpose of the conjugate of a matrix A is called transpose conjugate of A and is denoted by A^θ or A^*.
- **Properties of the Transpose Conjugate**
  (i) $(A^\theta)^\theta = A$
  (ii) $(A + B)^\theta = A^\theta + B^\theta$
  (iii) $(kA)^\theta = \bar{k}A^\theta$, k being any number
  (iv) $(AB)^\theta = B^\theta A^\theta$.
  (v) $(A^n)^\theta = (A^\theta)^n$.
- **Hermitian Matrix**
  A square matrix $A = [a_{ij}]$ is said to be hermitian matrix, if $a_{ij} = \bar{a}_{ji}$, $\forall\ i, j$ *i.e.*, $A = A^\theta$.
- **Properties of Hermitian Matrices**
  (i) If A is hermitian matrix, then kA is also hermitian for any real number k.
  (ii) If A and B be hermitian matrices of the same order, then $\lambda_1 A + \lambda_2 B$ is also hermitian for the real numbers (λ_1, λ_2)
  (iii) If A be any square matrix, then AA^* and A^*A are also hermitian.
  (iv) If A and B the hermitian matrices, then AB is also hermitian, iff $AB = BA$.
  (v) If A is a hermitian matrix, then $\bar{A}$ is also hermitian.
  (vi) If A and B are hermitian matrices of the same order, then $AB + BA$ is also hermitian matrix.
  (vii) If A is square matrix, then $A + A^*$ is a hermitian matrix.
  (viii) Any square matrix can be uniquely expressed as $A + iB$, where A and B are hermatian matrices.
- **Skew-Hermitian Matrix**
  A square matrix $A = [a_{ij}]$ is said to be a skew-hermitian matrix, $a_{ij} = -\bar{a}_{ji}$, $\forall i, j$ *i.e.*, $A^\theta = -A$.
- **Properties of a skew-hermitian matrix**
  (i) If A is a skew-hermitian matrix, then kA is also skew-hermitian for any real number k.
  (ii) If A and B be skew-hermitian matrices of the same order, then $\lambda_1 A + \lambda_2 B$ is also a skew-hermitian for the real numbers (λ_1, λ_2).
  (iii) If A and B are hermitian matrices of the same order, then $AB - BA$ is a skew-hermitian matrix.
  (iv) If A is any square matrix, then $A - A^*$ is a skew-hermitian matrix.
  (v) Every square matrix can be uniquely represented as the sum of a hermitian and a skew-hermitian matrix.
  (vi) If A is a skew-hermitian matrix, then iA is a hermitian matrix.
  (vii) If A is a skew-hermitian matrix, then $\bar{A}$ is also a skew-hermitian matrix.
- **Adjoint of a Square Matrix**
  Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be the cofactor of a_{ij} in the determinant A. Then the adjoint of A, denoted by adj A, is defined as the transpose of the cofactor matrix.
  The adjoint of a square matrix A is obtained by replacing each of the $(i, j)^{\text{th}}$ element of A by the cofactor of the $(j, i)^{\text{th}}$ element in $|A|$.
- **Properties of the Adjoint of a Matrix**
  (i) If A is a square matrix of order n, then $A(\text{adj } A) = |A| I_n = (\text{adj } A)A$, where I_n is a identity matrix of order n.

(ii) If A is square matrix of order n, then adj $(A') = (\text{adj } A)'$.

(iii) If A and B are two square matrices of the same order, then $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$.

(iv) $\text{adj}(\text{adj } A) = |A|^{n-2}A$, where A is a non-singular matrix.

(v) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$, where A is a non-singular matrix.

- **Inverse of a Square Matrix**

  Let A be any n-rowed square matrix. Then a square matrix B, such that $AB = BA = I$, is called inverse of A. The inverse of A deonted by A^{-1} and is determined by the formula

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

  It may be noted that $AA^{-1} = A^{-1}A = I$.

- **Properties of the Inverse of a Matrix**

  (i) A square matrix is invertible, if and only if it is non-singular.

  (ii) The inverse of the inverse is the original matrix itself *i.e.*, $(A^{-1})^{-1} = A$.

  (iii) The inverse of the transpose of a matrix is the transpose of its inverse *i.e.*, $(A')^{-1} = (A^{-1})'$.

  (iv) If A and B are two invertible matrices of the same order, then AB is also invertible and moreover $(AB)^{-1} = B^{-1}A^{-1}$.

  (v) Let A, B and C be square matrices of the same order n. If A is a non-singular matrix, then

  (a) $AB = AC \Rightarrow B = C$ (Left cancellation law)

  (b) $BA = CA \Rightarrow B = C$ (Right cancellation law)

  (vi) If A is a non-singular matrix such that A is symmetric, then A^{-1} is also symmetric.

  (vii) If A is a non-singular matrix, then $|A^{-1}| = |A|^{-1}$.

- **Rank of a Matrix**

  A positive integer r is said to be the rank of a non-zero matrix A, if

  (i) there exists at least one minor in A of order r which is not zero.

  (ii) every minor in A of order greater than r is zero. It is written as $\rho(A) = r$.

  The rank of a zero matrix is defined to be zero

- **Properties of Rank of a Matrix**

  (i) The rank of a null matrix is defined as zero *i.e.*, $\rho(O) = 0$.

  (ii) If I_n is a unit matrix of order n, then its rank $I_n = n$. *i.e.*, $\rho(I_n) = n$.

  (iii) From the definition of the rank of a matrix, we conclude that

  (a) If a matrix A does not possess any minor of order r, then $\rho(A) \geq r$

  (b) If at least one minor of order r of the matrix is not equal to zero, then $\rho(A) \leq r$.

  (iv) If every $(r + 1)$th order minor A is zero, then any higher order minor will also be zero.

  (v) If A is $n \times n$ non-singular matrix, the $\rho(A) = n$.

  (vi) Elementary operations do not change the rank of a matrix.

  (vii) If A' is a transpose of A, then $\rho(A') = \rho(A)$.

  (viii) If A^* is the transpose conjugate of A, then $\rho(A^*) = \rho(A)$.

  (ix) $\rho(A + B) \leq \rho(A) + \rho(B)$.

  (x) If A and B are two matrices such that the product AB is defined, then rank (AB) cannot exceed the rank of the either matrix.

- **Echelon Form of a Matrix**

  A non-zero matrix A is said to be in Echelon form if, A satisfies the following conditions :

  1. All the non-zero rows of A, if any preceed the zero rows.

2. The number of zeros preceeding the first non-zero element in a row is less than the number of such zeros in the succeeding row.
3. The first non-zero rows of a matrix given in the Echelon form is its rank.

- **Homogeneous Equations**

The system of equations $AX = B$ is said to be homogeneous if the constants $b_1, b_2, ..., b_n$ are all zero. That is, if the matrix B is a zero matrix and the system is of the form $AX = O$ where O is a null matrix of order $n \times 1$.

(i) If $|A| \neq 0$, then its only solution $X = O$, is called the trivial solution.

(ii) If $|A| = 0$, then $AX = O$ have a non-trivial solution. It will have infinitely many solutions.

- **Solutions of a system of linear equations by Matrix-Rank Method**

Let $AX = B$ be a system of n linear equations in n variables.

1. Write the augmented matrix $[A\ B]$.
2. Reduce the augmented matrix to echelon form using elementary row operations.
3. Determine the rank of the coefficient matrix A and augmented matrix [A B] by counting the number of non-zero rows in A and $[A\ B]$.

- **Characteristic equation of matrix**

$|A - \lambda I_n| = 0$ is called the characteristic equation of A. The values of λ are called 'eigen values' of A.

(i) The sum of the eigen values of A is equal to its trace.

(ii) The product of the eigen values of A is equal to its determinant.

(iii) The eigen values of an orthogonal matrix are of unit modulus.

(iv) The eigen values of a unitary matrix are of unit modulus.

(v) A and A' have same eigen values.

(vi) The eigen values of a skew-hermitian matrix are either purely imaginary or zero.

(vii) If λ is an eigen value of A, then λ is the eigen value of A^{θ}.

(viii) The eigen values of a triangular matrix are its diagonal elements.

(ix) If λ is the eigen value of A and $|A| \neq 0$, then $(1/\lambda)$ is the eigen value of A^{-1}.

(x) If λ is the eigen value of A and $|A| \neq 0$, then $\frac{|A|}{\lambda}$ is the eigen value of adj A.

(xi) If $\lambda_1, \lambda_2, ..., \lambda_n$ are eigen values of A, then the eigen values of A^2 are $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$.

- **Integration of determinants**

If $\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ \lambda_1 & \lambda_2 \end{vmatrix}$, then $\int_a^b \Delta(x)dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx \\ \lambda_1 & \lambda_2 \end{vmatrix}$

Here, $f(x)$ and $g(x)$ are functions of x and λ_1, λ_2 are constants.

- **Some particular determinants to remeber**

1. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

2. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

3. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)[(a^2+b^2+c^2)+(ab+bc+ca)]$

4. $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

5. $\begin{vmatrix} x+a & b & c & d \\ a & x+b & c & d \\ a & b & x+c & d \\ a & b & c & x+d \end{vmatrix} = x^3(x+a+b+c+d)$

6. $\begin{vmatrix} x^2 & (x+a)^2 & (x-a)^2 \\ y^2 & (y+a)^2 & (y-a)^2 \\ z^2 & (z+a)^2 & (z-a)^2 \end{vmatrix} = -4a^3(x-y)(y-z)(z-x)$

- **Maximum and minimum value of determinants when elements are known**

If, $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$ where $a_i's \in \{\alpha_1, \alpha_2, ..., \alpha_n\}$, then $|A|$ is maximum when, diagonal elements are $\{\min(\alpha_1, \alpha_2, ..., \alpha_n)\}$ and non-diagonal elements are $\{\max(\alpha_1, \alpha_2, ..., \alpha_n)\}$

Also, $|A|_{\min} = -|A|_{\max}$.

Miscellaneous Examples

1. If $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then show that, $F(x) \,.\, F(y) = F(x+y)$.Hence prove that $[F(x)]^{-1} = F(-x)$

Soln.: It is given that $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We have to show that, $F(x) \,.\, F(y) = F(x+y)$ and

to prove that $[F(x)]^{-1} = F(-x)$

From definition of product of two matrices, $A = [a_{ij}]_{m \times n}$ be a matrix of order $m \times n$ and $B = [b_{jk}]_{n \times p}$ be a matrix of order $n \times p$. Then the product AB is denoted as matrix $C = [c_{ik}]_{m \times p}$ of order $m \times p$, where, $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

$$F(x)\,F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos x\cos y - \sin x \sin y & -\cos x \sin y - \sin x \cos y & 0 \\ \sin x \cos y + \cos x \sin y & -\sin x \sin y + \cos x \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(x+y)$$

i.e., $F(x) \,.\, F(y) = F(x+y)$ proved

Replacing y by $-x$ in (1), we get, $F(x)\,F(-x) = F(x-x) = F(0) = \begin{bmatrix} \cos 0 & -\sin 0 & 0 \\ \sin 0 & \cos 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

i.e., $F(x)\,F(-x) = I \Rightarrow [F(x)]^{-1} = F(-x)$ proved

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2. Find the inverse of the matrix A, given by $A = \begin{bmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

Soln.: Given matrix $A = \begin{bmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

We have to determine A^{-1}

We know that $A^{-1} = \dfrac{(\text{adj } A)}{|A|}$

For given A

Co-factor $C_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$, $C_{12} = -\begin{vmatrix} -3 & 1 \\ -1 & 2 \end{vmatrix} = 5$

$C_{13} = \begin{vmatrix} -3 & 0 \\ -1 & 1 \end{vmatrix} = -3$, $C_{21} = -\begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix} = 6$

$C_{22} = \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 8$, $C_{23} = -\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = -1$

$C_{31} = \begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix} = -1$, $C_{32} = -\begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = -14$

$C_{33} = \begin{vmatrix} 2 & -1 \\ -3 & 0 \end{vmatrix} = -3$

$\therefore$ $\text{Adj}\, A = \begin{bmatrix} -1 & 6 & -1 \\ 5 & 8 & -14 \\ -3 & -1 & -3 \end{bmatrix}$ and $|A| = \begin{vmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{vmatrix} = 2(-1) + 1(-6+1) + 4(-3-0)$

$= -2 - 5 - 12 = -19$

$\therefore$ $A^{-1} = \dfrac{\text{adj}\, A}{|A|} = -\dfrac{1}{19}\begin{bmatrix} -1 & 6 & -1 \\ 5 & 8 & -14 \\ -3 & -1 & -3 \end{bmatrix}$

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3. Use matrix method to solve the following system of equations $x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 7z = 30$

Soln.: Given the system of equations $x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 7z = 30$

We have to solve the above system of equations to determine the value of x, y, z using matrix method.

For the system $AX = B$, if $|A| \neq 0$ then $X = A^{-1}B$

The system can be written in matrix equation $AX = B$ ...(1)

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 6 \end{vmatrix}$

By $C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1$

$= 1(6 - 6) = 0$

$\therefore$ the equations either has no solution or an infinite number of solutions. To decide about this, we proceed to find $(\text{Adj}\, A)\, B$.

Now, $C_{11} = 2$, $C_{12} = -4$, $C_{13} = 2$, $C_{21} = -3$, $C_{22} = 6$, $C_{23} = -3$, $C_{31} = 1$, $C_{32} = -2$, $C_{33} = 1$.

$\text{Adj}\, A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix}$ and $(\text{Adj}\, A)\, B = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$

Since $|A| = 0$ and $(\text{Adj}\, A)\, B = 0$, hence the equations are consistent and have infinitely many solutions.

To find these solutions, we proceed as follows:

We have

$[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 14 \\ 1 & 4 & 7 & \vdots & 30 \end{bmatrix} \Rightarrow R_{21}(-1) \sim R_{31}(-1) \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 3 & 6 & \vdots & 24 \end{bmatrix} \Rightarrow R_{32}(-3) \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$

$\therefore$ The equation (1) reduces to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} x + y + z \\ y + 2z \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$

$\therefore x + y + z = 6$, $y + 2z = 8$

Taking $z = k$, any real number, $y = 8 - 2k$, $x = k - 2$

Hence $x = k - 2$, $y = 8 - 2k$, $z = k$ where k is any real number, is the general solutions of the given equations.

Since k is arbitrary, hence the number of solutions are infinite.

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4. Show that the necessary and sufficient condition for the existence of the inverse of a square matrix A is that $|A| \neq 0$.

Soln.: A is a square matrix.

We have to show that $|A| \neq 0$ is a necessary and sufficient condition for the existence of the inverse of a square matrix A.

The condition is necessary.

i.e. if A is invertible then to show that $|A| \neq 0$

Let B be the inverse of the invertible matrix A, then $AB = BA = I_n$, where I_n is a unit matrix of the same order as A.

$\Rightarrow \quad |AB| = |BA| = I_n \Rightarrow |A|\ |B| = |B|\ |A| = 1 \neq 0$

$\therefore \quad |A| \neq 0$

i.e. the necessary condition for the existance of the inverse of the matrix A is that $|A| \neq 0$

The condition is sufficient.

i.e. if $|A| \neq 0$, then to show that A is invertible.

Consider the matrix B such that $B = \dfrac{1}{|A|}$ (Adj. A).

Then $AB = A\left(\dfrac{1}{|A|}\text{Adj } A\right) = \dfrac{1}{|A|}\{A \text{ Adj } A\} = \dfrac{1}{|A|}(|A|\ I_n) = I_n$

Also $BA = \left(\dfrac{1}{|A|}\text{Adj } A\right).A = \dfrac{1}{|A|}\{(\text{Adj } A).A\} = \dfrac{1}{|A|}|A|\ I_n = I_n$

Thus $AB = BA = I_n$

Hence the matrix A is invertible and inverse of A is given by $A^{-1} = B = \dfrac{\text{Adj } A}{|A|}$

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5. If A and B are symmetric matrices of the same size, then

(a) $AB + BA$ is symmetric

(b) $AB - BA$ is skew symmetric

Soln.: We have two symmetric matrices A and B of same order.

We have to show that (i) $AB + BA$ is symmetric and (ii) $AB - BA$ is skew-symmetric

Any matrix A is symmetric if $A^t = A$ and skew-symmetric if $A^t = -A$

(i) $(AB + BA)^t = (AB)^t + (BA)^t = B^t A^t + A^t B^t$

Now A and B are given to be symmetric,

$\therefore \quad A^t = A$ and $B^t = B \quad \therefore (AB + BA)^t = BA + AB = AB + BA$

$\therefore$ By definition $(AB + BA)$ is symmetric

(ii) $(AB - BA)^t = B^t A^t - A^t B^t = BA - AB = -(AB - BA)$

$\therefore \quad (AB - BA)^t = -(AB - BA)$

$\therefore$ By definition $(AB - BA)$ is skew-symmetric.

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6. If $u_n = \displaystyle\int_0^{\pi/2} \frac{1 - \cos 2nx}{1 - \cos 2x}dx$, evaluate $\Delta = \begin{vmatrix} \pi/2 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{vmatrix}$.

Soln.: It is given that $u_n = \displaystyle\int_0^{\pi/2} \frac{1 - \cos 2nx}{1 - \cos 2x}dx$

We have to find out the value of $\Delta = \begin{vmatrix} \pi/2 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{vmatrix}$

$$u_n + u_{n+2} - 2u_{n+1} = \int_0^{\pi/2} \frac{[1 - \cos 2nx + 1 - \cos(2n + 4)x - 2\{1 - \cos(2n + 2)x\}]}{(1 - \cos 2x)}dx$$

$$= \int_0^{\pi/2} \frac{2\cos(2n + 2)x - 2\cos(2n + 2)x \cos 2x}{(1 - \cos 2x)}dx$$

$= \int_0^{\pi/2} \frac{2\{\cos(2n+2)x\}(1-\cos 2x)}{(1-\cos 2x)} dx = \int_0^{\pi/2} 2\cos(2n+2)x\, dx$

$= 2\left[\frac{\sin(2n+2)x}{2n+2}\right]_0^{\pi/2} = 0$ ...(1)

$\therefore \quad u_n + u_{n+1} - 2u_{n+1} = 0$

Also $u_1 = \int_0^{\pi/2} \frac{1-\cos 2x}{1-\cos 2x} dx = \frac{\pi}{2}$ ...(2)

Therefore given determinant take the form, $\Delta = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{vmatrix}$

Now applying $C_1 \to C_1 + C_3 - 2C_2$ on given determinant, we get

$\Delta = \begin{vmatrix} u_1+u_3-2u_2 & u_2 & u_3 \\ u_4+u_6-2u_5 & u_5 & u_6 \\ u_7+u_9-2u_8 & u_8 & u_9 \end{vmatrix} = \begin{vmatrix} 0 & u_2 & u_3 \\ 0 & u_5 & u_6 \\ 0 & u_8 & u_9 \end{vmatrix} = 0$ (using (1) and (2))

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7. If $a = \cos\theta + i\sin\theta$, $b = \cos 2\theta - i\sin 2\theta$, $c = \cos 3\theta + i\sin 3\theta$ and if $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$ show that $\theta = 2k\pi$, $k \in Z$.

Soln.: It is given that $a = \cos\theta + i\sin\theta$, $b = \cos 2\theta - i\cos 2\theta$, $c = \cos 3\theta + i\sin 3\theta$ and also, $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

We have to show that, $\theta = 2k\pi$, $k \in Z$

We can solve this problem by simply expanding the given determinant and equating it zero.

$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2] = 0$

$\Rightarrow \quad a+b+c = 0$ or $a = b = c$

If $a+b+c = 0$, we have $\cos\theta + \cos 2\theta + \cos 3\theta = 0$

$\sin\theta - \sin 2\theta + \sin 3\theta = 0$

$\therefore \quad \cos 2\theta(2\cos\theta - 1) = 0$

and $\sin 2\theta(1 - 2\cos\theta) = 0$ ...(1)

Which is not possible as $\cos 2\theta = 0$ gives $\sin 2\theta \neq 0$, $\cos\theta \neq 1/2$

and $\cos\theta = 1/2$ gives $\sin 2\theta \neq 0$, $\cos\theta \neq 1/2$

$\therefore$ Equation (1) do not hold simultaneously $\therefore a+b+c \neq 0 \therefore a = b = c$ or $e^{i\theta} = e^{-2i\theta} = e^{3i\theta}$

which is satisfied only by $e^{i\theta} = 1$ i.e. $\cos\theta = 1$, $\sin\theta = 0$, so $\theta = 2k\pi$, $k \in Z$.

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8. Let $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$, evaluate $\lim_{x\to 0}[f'(x)/x]$.

Soln.: $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$

To find out the value of $\lim_{x\to 0}[f'(x)/x]$

We will try to simplify the given determinant by applying elementary row and column operations.

$$f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$, we get $f(x) = \begin{vmatrix} \cos x - \tan x & 0 & 0 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$

Expanding along R, we get $f(x) = (\cos x - \tan x)(x^2 - 2x^2) = -x^2(\cos x - \tan x)$

$\therefore\ f'(x) = -2x(\cos x - \tan x) - x^2(-\sin x - \sec^2 x)$

$$\lim_{x\to 0} \frac{f'(x)}{x} = \lim_{x\to 0} \{-2(\cos x - \tan x) + x(\sin x + \sec^2 x)\} = -2 \times 1 = -2.$$

---

9. Sum the series $\sum_{r=2}^{n} (-2)^r \begin{vmatrix} {}^{n-2}C_{r-2} & {}^{n-2}C_{r-1} & {}^{n-2}C_r \\ -3 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix}$, for $n > 2$.

Soln.: The given determinant is $\Delta = \sum_{r=2}^{n} (-2)^r \begin{vmatrix} {}^{n-2}C_{r-2} & {}^{n-2}C_{r-1} & {}^{n-2}C_r \\ -3 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix}$

Applying $C_1 \to C_1 + 2C_2 + C_3$

$$S = \sum_{r=2}^{n} (-2)^r \begin{vmatrix} {}^{n}C_r & {}^{n-2}C_{r-1} & {}^{n-2}C_r \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix} = \sum_{r=2}^{n} (-2)^r\ {}^{n}C_r = \sum_{r=0}^{n} (-2)^r\ {}^{n}C_r - ({}^{n}C_0 - 2\,{}^{n}C_1)$$

$= (1 - 2)^n - (1 - 2n) = 2n - 1 + (-1)^n.$

---

10. If $\Delta_r = \begin{vmatrix} 2^{r-1} & \dfrac{(r+1)!}{(1+1/r)} & 2r \\ a & b & c \\ 2^n - 1 & (n+1)! - 1 & n(n+1) \end{vmatrix}$, then evaluate the value of $\sum_{r=1}^{n} \Delta_r$.

Soln.: A determinant $\Delta_r = \begin{vmatrix} 2^{r-1} & \dfrac{(r+1)!}{(1+1/r)} & 2r \\ a & b & c \\ 2^n - 1 & (n+1)! - 1 & n(n+1) \end{vmatrix}$

We have to evaluate the value of $\sum_{r=1}^{n} \Delta_r$.

We know that if in a determinant two rows or two columns are same then the determinant is equal to zero.

$$\text{We have } \sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} 2^{r-1} & \sum_{r=1}^{n} \dfrac{(r+1)!}{(1+1/r)} & 2\sum_{r=1}^{n} r \\ a & b & c \\ 2^n - 1 & (n+1)! - 1 & n(n+1) \end{vmatrix}$$

But $\sum_{r=1}^{n} 2^{r-1} = 2^n - 1,\ 2\sum_{r=1}^{n} r = 2\left\{\frac{n}{2}(n+1)\right\} = n(n+1)$

$$\sum_{r=1}^{n} \frac{(r+1)!}{(1+1/r)} = \sum_{r=1}^{n} r(r!) = \sum_{r=1}^{n} (r+1-1)(r!) = \sum_{r=1}^{n} \{(r+1)! - r!\} = (n+1)! - 1$$

$$\sum_{r=1}^{n}\Delta_r = \begin{vmatrix} 2^n-1 & (n+1)!-1 & n(n+1) \\ a & b & c \\ 2^n-1 & (n+1)!-1 & n(n+1) \end{vmatrix} \quad \because R_1 \text{ and } R_3 \text{ are identical}$$

$$\therefore \sum_{r=1}^{n}\Delta_r = 0.$$

---

11. If $A = \begin{bmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{bmatrix}$, find det A.

Soln.: $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ gives

$$\Delta = \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}(a+b+c)^2 \qquad R_1 \to R_1 - (R_2 + R_3) \text{ gives}$$

$$\Delta = 2\begin{vmatrix} bc & -c & -b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}(a+b+c)^2$$

$C_2 \to C_2 + \frac{1}{b}C_1$, $C_3 \to C_3 + \frac{1}{c}C_1$ gives

$$\Delta = 2\begin{vmatrix} bc & 0 & 0 \\ b^2 & c+a & +\frac{b^2}{c} \\ c^2 & +\frac{c^2}{b} & a+b \end{vmatrix}(a+b+c)^2 = 2bc\ [(c+a)(a+b) - bc]\ (a+b+c)^2$$

$$= 2abc\ (a+b+c)^3.$$

---

12. If $a, b, c \in R$ and not all zero then show that $\begin{vmatrix} ax-by-cz & ay+bx & cx+az \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix} = 0$

represent either equation of plane excluding a particular point or represent that point only or plane including that point.

Soln.: We have, $\begin{vmatrix} ax-by-cz & ay+bx & cx+az \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix} = 0$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} a^2x-aby-acz & ay+bx & cx+az \\ a^2y+abx & by-cz-ax & bz+cy \\ acx+a^2z & bz+cy & cz-ax-by \end{vmatrix} = 0 \quad (C_1 \to C_1 + bC_2 + cC_3)$$

$$\Rightarrow (a^2+b^2+c^2)\frac{1}{a}\begin{vmatrix} x & ay+bx & cx+az \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix} = 0 \quad \Rightarrow (a^2+b^2+c^2)\frac{1}{ax}\begin{vmatrix} x^2 & bxy+bx^2 & cx^2+axz \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix} = 0$$

$R_1 \to R_1 + yR_2 + zR_3$

$$\Rightarrow (a^2+b^2+c^2)(x^2+y^2+z^2)\frac{1}{ax}\begin{vmatrix} 1 & b & c \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix} = 0$$

$R_2 \to R_2 - yR_1$ and $R_3 \to R_3 - zR_1$

$$\Rightarrow (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)\frac{1}{ax}\begin{vmatrix} 1 & b & c \\ 0 & -cz-ax & bz \\ 0 & cy & -ax-by \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{ax}(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)(acxz + a^2x^2 + bcyz + abxy - bcyz) = 0$$

$\Rightarrow (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)(ax + by + cz) = 0$

$\Rightarrow$ either $x^2 + y^2 + z^2 = 0$ or $ax + by + cz = 0$

If $(x^2 + y^2 + z^2 \neq 0)$ or both zero

$\Rightarrow$ represent sphere with radius zero i.e. point origin only or representing plane excluding origin or plane containing origin.

---

13. If $\Delta \neq 0$ and Δ^c denotes the determinant of co-factors then $\Delta^c = \Delta^{n-1}$, where $n(> 0)$ is the order of Δ. In particular Δ is of order 3 then $\Delta^c = \Delta^2$.

Soln.: Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$ then $\Delta^c = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$

we prove the theorem for $n = 3$

we know that $\sum a_{ip}\, c_{jp} = \Delta$ if $i = j = 0$ if $i \neq j$

Now $\Delta \,.\, \Delta^c = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \times \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$

$= \begin{vmatrix} \sum a_{1p}C_{1p} & \sum a_{1p}C_{2p} & \sum a_{1p}C_{3p} \\ \sum a_{2p}C_{1p} & \sum a_{2p}C_{2p} & \sum a_{2p}C_{3p} \\ \sum a_{3p}C_{1p} & \sum a_{3p}C_{2p} & \sum a_{3p}C_{3p} \end{vmatrix}$ (using row × row multiplication)

$\therefore \quad \Delta \,.\, \Delta^c = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$

As $\Delta \neq 0$, $\therefore \Delta^c = \Delta^2$.

---

14. Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{bmatrix}$ and $B = A^2$

If $(a - b)^2 + (p - q)^2 = 25$

$(b - c)^2 + (q - r)^2 = 36$

$(c - a)^2 + (r - p)^2 = 47$

then find det B.

Soln.: det A is twice the area of the triangle with vertices (a, p), (b, q), (c, r) with sides 5, 6, 7

$\Delta^2 = s\,(s - a)\,(s - b)\,(s - c)$

$16\Delta^2 = 18 \cdot 8 \cdot 6 \cdot 4$

det $B = (\det A)^2 = 4\Delta^2 = 18 \cdot 8 \cdot 6 = 864$.

---

15. Let a, b, c be positive and not all equal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative

Soln.: Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Expanding $\Delta = a\,(bc - a^2) - b\,(b^2 - ac) + c\,(ab - c^2) = -[a^3 + b^3 + c^3 - 3abc]$

$= -\left[\frac{1}{2}(a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)\right]$

$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] < 0$ $\quad$ ($\because$ a, b, c are +ve and not all equal)

---

16. Without expanding a determinant at any stage, show that

$\begin{vmatrix} x^2 + x & x+1 & x-2 \\ 2x^2 + 3x - 1 & 3x & 3x-3 \\ x^2 + 2x + 3 & 2x-1 & 2x-1 \end{vmatrix} = xA + B$ where A and B are determinants of order 3 not involving x.

Soln.: Given a determinant $\begin{vmatrix} x^2 + x & x+1 & x-2 \\ 2x^2 + 3x - 1 & 3x & 3x-3 \\ x^2 + 2x + 3 & 2x-1 & 2x-1 \end{vmatrix}$

We have to determine two determinants A, B of order 3 not involving x such that the given determinant $= Ax + B$.

The procedure will involve making as many as possible entries to be zero by using the elementary row and column operations.

$$\begin{vmatrix} x^2 + x & x+1 & x-2 \\ 2x^2 + 3x - 1 & 3x & 3x-3 \\ x^2 + 2x + 3 & 2x-1 & 2x-1 \end{vmatrix} \quad (\text{Apply } R_1 \to R_1 + R_3)$$

$$= \begin{vmatrix} 2x^2 + 3x + 3 & 3x & 3x-3 \\ 2x^2 + 3x - 1 & 3x & 3x-3 \\ x^2 + 2x + 3 & 2x-1 & 2x-1 \end{vmatrix} \quad (\text{By } R_1 \to R_1 - R_2)$$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 2x^2 + 3x - 1 & 3x & 3x-3 \\ x^2 + 2x + 3 & 2x-1 & 2x-1 \end{vmatrix} \quad (\text{By } C_2 \to C_2 - C_3)$$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 2x^2 + 3x - 1 & 3 & 3x-3 \\ x^2 + 2x + 3 & 0 & 2x-1 \end{vmatrix} \quad (\text{By } C_1 \to C_1 - C_3)$$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 2x^2 + 2 & 3 & 3x-3 \\ x^2 + 4 & 0 & 2x-1 \end{vmatrix}$$

By $R_2 \to R_2 - \left(\frac{x^2}{2}\right)R_1$ and $R_3 \to R_3 - \left(\frac{x^2}{4}\right)R_1$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3x-3 \\ 4 & 0 & 2x-1 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3x \\ 4 & 0 & 2x \end{vmatrix} + \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix} = x\begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3 \\ 4 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix}$$

Thus $A = \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3 \\ 4 & 0 & 2 \end{vmatrix}$, $B = \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix}$ are of order 3 and independent of

17. Show that the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + lz = -3$ has at least one solution for any real number l ¹ –5. Find the set of solutions if l = –5.

Soln.: $D = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} = 3\,(2\lambda + 15) + (\lambda + 18) + 4\,(5 - 12) = 7\,(\lambda + 5)$

For unique solution $D \neq 0 \quad \therefore \lambda + 5 \neq 0 \quad \therefore \lambda \neq -5$

If $\lambda = -5$, $D = 0$ then $D_1 = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ -3 & 5 & -5 \end{vmatrix} = 3\,(-10 + 15) + 1\,(10 - 9) + 4\,(-10 + 6) = 15 + 1 - 16 = 0$

$$D_2 = \begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & 5 \end{vmatrix} = 3(10-9) - 3(-5+18) + 4(-3+12) = 3 - 39 + 36 = 0$$

$$D_3 = \begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & 5 & -3 \end{vmatrix} = 3(-6+10) + 1(-3+12) + 3(5-12) = 12 + 9 - 21 = 0$$

$\therefore$ The system has infinite number of solutions.

Let $z = k$. then the equations become $3x - y = 3 - 4k$, $x + 2y = 3k - 2$.

Solving, $x = \dfrac{4-5k}{7}, y = \dfrac{13k-9}{7}, z = k, k \in \mathbb{R}$

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18. Let α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$, $C(x)$ be polynomials of degree 3, 4 and 5 respectively. Then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$ where dash denotes the derivatives

Soln.: Clearly, since α is a repeated root of the quadratic equation $f(x) = 0$, therefore $f(x) = a\,(x - \alpha)^2$ where $(a \neq 0)$

Let $g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} \quad \therefore g(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$ since rows 1 and 2 are identical

Again $\therefore g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} \quad \therefore g'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$ since rows 1 & 3 are identical

Thus α is repeated root of $g(x) = 0$ $\therefore g(x)$ is divisible by $(x - \alpha)^2$ $\therefore g(x)$ is divisible by $f(x)$.

---

19. Show that $\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix} = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$

Soln.: Let $\Delta = \begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix}$

Operating $C_3 \to C_3 + C_2$, then $C_2 \to C_2 + C_1$ and using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$$\Delta = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$$

Again operating $C_3 \to C_3 + C_2$

$$\Delta = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$

---

20. Let $\Delta a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$. Show that $\sum_{a=1}^{n} \Delta a = c,$ a constant.

Soln.: Here $\Delta a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

Taking summation over a $\sum_{a=1}^{n} \Delta a = \begin{vmatrix} \sum_{a=1}^{n}(a-1) & n & 6 \\ \sum_{a=1}^{n}(a-1)^2 & 2n^2 & 4n-2 \\ \sum_{a=1}^{n}(a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

$= \frac{n(n-1)}{12}\begin{vmatrix} 6 & n & 6 \\ 4n-2 & 2n^2 & 4n-2 \\ 3n^2-3n & 3n^3 & 3n^2-3n \end{vmatrix} = \frac{n(n-1)}{12} \times 0 = 0$ ($\because$ two column's are identical)

= 0, a constant.

---

21. Let the three digit numbers $A28$, $3B9$, and $62C$, where A, B and C are integers between 0 and 9, be divisible by a fixed integer k. Show that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is divisible by k.

Soln.: $A28 = 100A + 20 + 8 = ak$ .... (1)

$3B9 = 300 + 10B + 9 = bk$ .... (2)

$62C = 600 + 20 + c = dk$ .... (3)

Where $A28$, $3B9$, $62C$ are divisible by k and a, b, d are suitable positive integer.

Now Det = $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$

$R_2 \to 100R_1 + 10R_3 + R_2$, Det $= \begin{vmatrix} A & 3 & 6 \\ A28 & 3B9 & 62C \\ 2 & B & 2 \end{vmatrix}$ (Using (1), (2), (3))

$= \begin{vmatrix} A & 3 & 6 \\ ak & bk & dk \\ 2 & B & 2 \end{vmatrix} = k\begin{vmatrix} A & 3 & 6 \\ a & b & d \\ 2 & B & 2 \end{vmatrix}$ is divisible by k.

---

22. If $a \neq p$, $b \neq q$, $c \neq$ r and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, then find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$.

Soln.: $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$

Applying the operations, $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$ gives

$$\begin{vmatrix} p-a & 0 & c-r \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

On solving this determinant we get $= (q-b)\,[(p-a)\,r - a\,(c-r)] - [(p-a)\,(c-r) - 0] \times b$

$= (q-b)\,(p-a)\,r - (q-b)\,(c-r)\,a + (p-a)\,(c-r)\,b$

Now $b = (q - (q-b))$ and $a = (p - (p-a))$

$= (q-b)\,(p-a)\,r + (q-b)\,(r-c)\,(p-p+a) + (p-a)\,(r-c)\,(q-q+b) = 0$

$= (q-b)\,(p-a)\,r + (q-b)\,(r-c)\,(p) - (p-a)\,(q-b)\,(r-c)$

$+ (p-a)\,(r-c)\,(q) - (q-b)\,(p-a)\,(r-c) = 0$

Now dividing throughout by $(p-a)\,(q-b)\,(r-c)$

We get $\frac{r}{r-c} + \frac{p}{p-a} - 1 + \frac{q}{q-b} - 1 = 0 \Rightarrow \frac{r}{r-c} + \frac{p}{p-a} + \frac{q}{q-b} = 2$

---

23. For a fixed positive integer n, if $\Delta = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$; then show that $\left[\frac{\Delta}{(n!)^3} - 4\right]$ is divisible by n.

Soln.: For a fixed positive integer n

$$\Delta = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

We have to prove that $\left[\frac{\Delta}{(n!)^3} - 4\right]$ is divisible by n

We will try to simplify the given determinant by elementary row and column operation and then expand it in order to get required result.

The given determinant is $\Delta = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$

Taking $n!$, $(n+1)!$, $(n+2)!$ Common from R_1, R_2 and R_3 respectively, we get

$$\Delta = n!\,(n+1)!\,(n+2)! \begin{vmatrix} 1 & (n+1) & (n+2)(n+1) \\ 1 & (n+2) & (n+3)(n+2) \\ 1 & (n+3) & (n+4)(n+3) \end{vmatrix}$$

$$\therefore \quad \frac{\Delta}{(n!)^3} = \frac{(n+1)!(n+2)!}{(n!)^2} \begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 1 & (n+2) & (n+2)(n+3) \\ 1 & (n+3) & (n+3)(n+4) \end{vmatrix}$$

$$= (n+1)^2 (n+2) \begin{vmatrix} 1 & n+1 & (n+1)(n+2) \\ 0 & 1 & 2(n+2) \\ 0 & 1 & 2(n+3) \end{vmatrix} \qquad (R_2 \to R_2 - R_1, R_3 \to R_3 - R_2)$$

$= (n + 1)^2 (n + 2) . 2 = 2[n^3 + 4n^2 + 5n + 2]$

$\therefore \quad \dfrac{\Delta}{(n!)^3} - 4 = 2n(n^2 + 4n + 5)$

Hence $\left(\dfrac{\Delta}{(n!)^3} - 4\right)$ is divisible by n.

---

24. Let λ and α be real. Find the set of all values of λ for which the system of linear equations $\lambda x + (\sin \alpha)y + (\cos \alpha)z = 0$, $x + (\cos \alpha)y + (\sin \alpha)z = 0$, $-x + (\sin \alpha)y - (\cos \alpha)z = 0$ has a non trivial solution. For $\lambda = 1$ find all values of α.

Soln.: Given λ and α be two real numbers.

Also $\lambda x + (\sin \alpha)y + (\cos \alpha)z = 0$

$x + (\cos \alpha)y + (\sin \alpha)z = 0$

$-x + (\sin \alpha)y - (\cos \alpha)z = 0$ has a non trivial solution.

We have to determine λ for which the system has non trivial solution.

Also for $\lambda = 1$ we have to find value of α.

The system of homogeneous equations has the non trivial solution if $\Delta = 0$.

System has non-trivial solution so, $\Delta = \begin{vmatrix} \lambda & \sin \alpha & \cos \alpha \\ 1 & \cos \alpha & \sin \alpha \\ -1 & \sin \alpha & -\cos \alpha \end{vmatrix} = 0$

$\therefore \quad \lambda[-\cos^2 \alpha - \sin^2 \alpha] - [-\sin \alpha \cos \alpha - \sin \alpha \cos \alpha] - [\sin^2 \alpha - \cos^2 \alpha] = 0$

or $-\lambda + \sin 2\alpha + \cos 2\alpha = 0$

$\Rightarrow \lambda = \sin 2\alpha + \cos 2\alpha = \sqrt{2} \cos(2\alpha - \pi/4)$ ...(1)

Since $-1 \le \cos(2\alpha - \pi/4) \le 1 \quad \therefore -\sqrt{2} \le \lambda \le \sqrt{2}$

$\Rightarrow \lambda \in [-\sqrt{2}, \sqrt{2}]$ ...(2)

Now for $\lambda = 1$

$\cos(2\alpha - \pi/4) = 1/\sqrt{2} \quad \Rightarrow \quad 2\alpha - \pi/4 = 2n\pi \pm \pi/4$

or $2\alpha = 2n\pi$ or $2n\pi + \pi/2$

$\therefore \quad \alpha = n\pi$ or $n\pi + \pi/4 \;\; n \in Z$.

---

25. For all A, B, C, P, Q, R, show that $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$

Soln.: Expanding Ist column using $\cos(A - B)$

Given determinant $= \cos P \begin{vmatrix} \cos A & \cos(A-Q) & \cos(A-R) \\ \cos B & \cos(B-Q) & \cos(B-R) \\ \cos C & \cos(C-Q) & \cos(C-R) \end{vmatrix} + \sin P \begin{vmatrix} \sin A & \cos(A-Q) & \cos(A-R) \\ \sin B & \cos(B-Q) & \cos(B-R) \\ \sin C & \cos(C-Q) & \cos(C-R) \end{vmatrix}$

Now $C_2 \to C_2 - C_1 \cos Q$ & $C_3 \to C_3 - C_1 \cos R$

Determinant $= \cos P \sin Q \sin R \begin{vmatrix} \cos A & \sin A & \sin A \\ \cos B & \sin B & \sin B \\ \cos C & \sin C & \sin C \end{vmatrix} + \sin P \cos Q \cos R \begin{vmatrix} \sin A & \cos A & \cos A \\ \sin B & \cos B & \cos B \\ \sin C & \cos C & \cos C \end{vmatrix} = 0$

(Two columns of a determinant are identical).

26. For $a > 0$, $d > 0$ find the value of the determinant $\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$

Soln.: Determinant $= \dfrac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)} \times \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix}$

As $\Delta = \begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix} = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)2d & d & d \\ (a+3d)2d & d & d \end{vmatrix}$ (Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$)

On expanding, $\Delta = 2d^2\,[(a+2d)\,d - ad] = 4d^4$

$\Rightarrow$ Value of given determinant is $\dfrac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$.

---

27. Find the value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ where a, b and c are respectively the p^{th}, q^{th} and r^{th} terms of a harmonic progression.

Soln.: $\because \Delta = \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ $\because$ a, b, and c are in H.P., the p^{th}, q^{th}, r^{th} term respectively

$\frac{1}{a} = A + (p-1)D$, $\frac{1}{b} = A + (q-1)D$, $\frac{1}{c} = A + (r-1)D$

where A be the first term and D be common difference

$\therefore \Delta = abc\begin{vmatrix} A+(p-1)D & A+(q-1)D & A+(r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

Operating $R_1 \to R_1 - (R_2 - R_3)D$ we get $abc\begin{vmatrix} A & A & A \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abcA\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$

$\because$ two rows are identical.

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28. Prove that for all values of θ, $\begin{vmatrix} \sin\theta & \cos\theta & \sin\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta - \frac{4\pi}{3}\right) \end{vmatrix} = 0$

Soln.: Denote the determinant on the LHS by Δ applying $R_2 \to R_2 + R_3$ and using
$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$
and $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

we get $\Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin\theta \\ 2\sin\theta\cos\left(\frac{2\pi}{3}\right) & 2\cos\theta\cos\left(\frac{2\pi}{3}\right) & 2\sin\theta\cos\left(\frac{4\pi}{3}\right) \\ 2\sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta-\frac{4\pi}{3}\right) \end{vmatrix}$

But $\cos\left(\frac{2\pi}{3}\right) = \cos\left(\pi-\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$ and $\cos\left(\frac{4\pi}{3}\right) = \cos\left(\pi+\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$

therefore R_1 and R_2 are identical.

Thus, $\Delta = 0 \ \forall\ \theta \in R$.

---

29. Let a,b,c are real numbers with $a^2 + b^2 + c^2 = 1$. Show that the equation represents a straight line.

$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0.$ **(2001)**

Soln.: Let us denote the given determinant by Δ, then using $C_1 \to aC_1$, we can write Δ as

$$\Delta = \frac{1}{a}\begin{vmatrix} a^2x-aby-ac & bx+ay & cx+a \\ abx+a^2y & -ax+by-c & cy+b \\ acx+a^2 & cy+b & -ax-by+c \end{vmatrix}$$

Applying $C_1 \to C_1 + b\,C_2 + c\,C_3$ we get $\Delta = \frac{1}{a}\begin{vmatrix} (a^2+b^2+c^2)x & bx+ay & cx+a \\ (a^2+b^2+c^2)y & -ax+by-c & cy+b \\ a^2+b^2+c^2 & cy+b & -ax-by+c \end{vmatrix}$

$$\Delta = \frac{1}{a}\begin{vmatrix} x & bx+ay & cx+a \\ y & -ax+by-c & cy+b \\ 1 & cy+b & -ax-by+c \end{vmatrix} \quad (\because a^2+b^2+c^2=1)$$

Alpplying $C_2 \to C_2 - b\,C_1$, $C_3 \to C_3 - c\,C_1$, we get $\Delta = \frac{1}{a}\begin{vmatrix} x & ay & a \\ y & -ax-c & b \\ 1 & cy & -ax-by \end{vmatrix}$

Multiplying R_1 by x, we get $\Delta = \frac{1}{ax}\begin{vmatrix} x^2 & axy & -ax \\ y & -ax-c & b \\ 1 & cy & -ax-by \end{vmatrix}$

Applying $R_1 \to R_1 + yR_2 + R_3$, we get $\Delta = \frac{1}{ax}\begin{vmatrix} x^2+y^2+1 & 0 & 0 \\ y & -ax-c & b \\ 1 & cy & -ax-by \end{vmatrix}$

Expanding along R_1, we get $\Delta = \frac{1}{ax}(x^2+y^2+1)\begin{vmatrix} -(ax+c) & b \\ cy & -(ax+by) \end{vmatrix}$

$= \frac{1}{ax}(x^2+y^2+1)\{(ax+c)(ax+by)-bcy\} = \frac{1}{ax}(x^2+y^2+1)\{ax(ax+by)+c(ax+by)-bcy\}$

$= \frac{1}{ax}(x^2+y^2+1)\{ax(ax+by)+c(ax)\} = (x^2 + y^2 + 1)\,(ax + by + c)$

Thus $\Delta = 0$. $\Rightarrow (x^2 + y^2 + 1)\,(ax + by + c) = 0$

As $x^2 + y^2 + 1 > 0$ we get $\Delta = 0 \Rightarrow ax + by + c = 0$ which represents a straight line.

---

30. If matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ where a, b, c are real positive numbers, $abc = 1$ and $A^T A = I$, then find the value of $a^3 + b^3 + c^3$. **(2003)**

Soln.: $A^TA = I$

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow\ a^2 + b^2 + c^2 = 1$ .... (1)

and $ab + bc + ca = 0$ .... (2)

Now $a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) + 3abc$

$= (a + b + c) + 3$ .....(3)

Now $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) = 1 + 2\cdot 0 = 1$

$\Rightarrow\ a + b + c = 1$ (since a, b, c are real positive number)

Now from (3) $a^3 + b^3 + c^3 = 1 + 3 = 4$

Aliter : $A^T A = 1 \Rightarrow |A^T A| = |1| \Rightarrow |A|^2 = 1 \Rightarrow (a^3 + b^3 + c^3 - 3abc)^2 = 1$

$\Rightarrow\ a^3 + b^3 + c^3 - 3abc = 1$ (since a, b, c are positive real number

$\Rightarrow a^3 + b^3 + c^3 \geq 3abc$ from $AM \geq GM$) $\Rightarrow a^3 + b^3 + c^3 \geq 3 \Rightarrow a^3 + b^3 + c^3 = 4$

---

31. If M is a 3×3 matrix, where det $(M) = 1$ and $MM^T = I$, then prove that det $(M - I) = 0$ **(2004)**

Soln.: For any matrix M there exist a real number λ and a non-zero column vector X such that $(M - \lambda I)X = O$

$\Rightarrow\ MX = \lambda IX \Rightarrow MX = \lambda X$ .... (1)

taking transpose of both side of (1)

$X^TM^T = \lambda X^T$ .... (2)

Multiplying (1) and (2)

$X^T M^T MX = \lambda^2 X^T X$ as $(M^T M = I)$ $\Rightarrow X^T IX = \lambda^2 X^T X \Rightarrow X^T X = \lambda^2 X^T X$ as X is a non zero column vector $X^T X \neq 0$

$\Rightarrow\ \lambda^2 = 1 \Rightarrow \lambda = 1$ or -1 $\therefore$ when $\lambda = 1$

$(M - I)X = 0 \Rightarrow |(M - I)X| = 0 \Rightarrow |M - I| = 0$

Aliter : Let $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & t \end{bmatrix}$

Since $M M^T = I$ $\therefore$ M^T is inverse of M

$\Rightarrow\ M^T$ is adjoint of M [Since det $(M) = I$]

$$\Rightarrow \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & t \end{bmatrix} = \begin{bmatrix} et - fh & ch - bt & bf - ce \\ gf - dt & at - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix} \quad .... (1)$$

Now $|M - I| = \begin{vmatrix} a-1 & b & c \\ d & e-1 & f \\ g & h & t-1 \end{vmatrix}$

On expanding above determinant and using (1)

$|M - I| = 0$

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32. $A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}, B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}, U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}, V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. If $AX = U$ has infinitely many solutions, prove that $BX = V$ has no unique solution, also prove that if $afd \neq 0$, then $BX = V$ has no solution. **(2004)**

Soln.: $AX = U \quad \Rightarrow \begin{bmatrix} a & 1 & 0 & : & x \\ 1 & b & d & : & y \\ 1 & b & c & : & z \end{bmatrix} = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 1-ab & 0-ad & : & x \\ 1 & b & d & : & y \\ 0 & 0 & c-d & : & z \end{bmatrix} = \begin{bmatrix} f-ag \\ g \\ h-g \end{bmatrix}$ as $AX = U$ has infinitely many solution

$(R_1 \to R_1 - aR_2, R_3 \to R_3 - R_2)$

$\Rightarrow d = c$ and $h = g$

Now $BX = V \;\Rightarrow \begin{bmatrix} a & 1 & 1 & : & x \\ 0 & d & c & : & y \\ f & g & h & : & z \end{bmatrix} = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a & 1 & 1 & : & x \\ 0 & d & c & : & y \\ 0 & 0 & h-\frac{f}{a}-\frac{c}{d}\left(g-\frac{f}{a}\right) & : & z \end{bmatrix} = \begin{bmatrix} a^2 \\ 0 \\ -af \end{bmatrix} \qquad \ldots (1)$$

If $c = d = 0$ and $h = g$ clearly $BX = V$ has no unique solution if $c = d \neq 0,\ h-\frac{f}{a}-\frac{c}{d}\left(g-\frac{f}{a}\right)=0$

$\Rightarrow$ $[A{:}X] = 0 \Rightarrow$ $BX = V$ has no unique solution

Now as $afd \neq 0$

$\Rightarrow$ $a \neq 0, f \neq 0, d \neq 0$ $\Rightarrow A : X] = 0$ but as $-af \neq 0$ $\therefore$ from equation (1)

$BX = V$ has no solution.

EXERCISE

MCQs One Correct Option

1. If $\begin{vmatrix} a+x & a & x \\ a-x & a & x \\ a-x & a & -x \end{vmatrix} = 0$, then x is

(a) 0 (b) a (c) 3 (d) $2a$.

2. $\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix}$ is equal to

(a) $p+q+r$ (b) 0
(c) $p-q-r$ (d) $-p+q+r$.

3. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then

(a) $x = 3, y = 1$ (b) $x = 1, y = 3$
(c) $x = 0, y = 3$ (d) $x = 0, y = 0$.

4. If x, y, z are integers in A.P., lying between 1 and 9, and $x51$, $y41$ and $z31$ are three digit numbers then the value of $\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$ is

(a) $x+y+z$ (b) $x-y+z$
(c) 0 (d) none of these.

5. Two non-zero distinct numbers a, b are used as elements to make determinants of third order. The number of determinants whose value is zero for all a, b is

(a) 24 (b) 32
(c) $a+b$ (d) none of these.

6. The sum of two non-integral roots of $\begin{vmatrix} x & 2 & 3 \\ 3 & x & 3 \\ 5 & 4 & x \end{vmatrix}$ is

(a) 5 (b) –5
(c) –18 (d) none of these.

7. If the determinant $\begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$ is expanded in powers of $\sin x$, then the constant term in the expansion is

(a) 1 (b) 2
(c) –1 (d) none of these.

8. The equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + mz = n$ give infinite number of values of the triplet (x, y, z) if

(a) $m = 3, n \in R$ (b) $m = 3, n \neq 10$
(c) $m = 3, n = 10$ (d) none of these.

9. The system of equations $2x + 3y = 8$, $7x - 5y + 3 = 0$, $4x - 6y + \lambda = 0$ is solvable if λ is

(a) 6 (b) 8 (c) -8 (d) -6

10. If $\Delta = \begin{vmatrix} \cos\theta/2 & 1 & 1 \\ 1 & \cos\theta/2 & -\cos\theta/2 \\ -\cos\theta/2 & 1 & -1 \end{vmatrix}$ lies in the interval

(a) [2, 4] (b) [0, 4] (c) [1, 3] (d) [–2, 2]

11. If $z = \begin{vmatrix} 2+5i & 2-3i & 4+i \\ 8i & 3i & -5i \\ 2-5i & 2+3i & 4-i \end{vmatrix}$, then

(a) both $Re\,(z)$ and $Im\,(z)$ are non-zero
(b) z is purely real
(c) z is purely imaginary
(d) $z = 0$.

12. If a, b, c are the sides of a triangle ABC such that $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0$, then

(a) ΔABC is a non-isosceles right-angled triangle
(b) ΔABC is an equilateral triangle
(c) ΔABC is an acute angled triangle with no two angles being equal
(d) ΔABC is an isosceles triangle.

13. If $x \in R$,

$f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & x(x^2-1) \end{vmatrix}$, then $f(27) =$

(a) 0 (b) –27 (c) 27 (d) 1.

14. If the system of equations $-ax + y + z = 0$, $x - by + z = 0$ and $x + y - cz = 0$ $(a, b, c \neq 1)$ has a non-zero solution, then $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} =$

(a) 0 (b) 1 (c) 2 (d) 3.

15. If $D_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+2 & n^2+n \\ 2k-1 & n^2 & n^2+n+2 \end{vmatrix}$ and $\sum_{k=1}^{n} D_k = 48$, then n equals

(a) 4 (b) 6
(c) 8 (d) none of these.

16. If $\begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix}$ $= (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$, then

(a) $k = -2$ (b) $k = -1$ (c) $k = 0$ (d) $k = 1$

17. If $\begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix} = f(x) - xf'(x)$ then $f(x)$, is equal to

(a) $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$
(b) $(x+\alpha)(x+\beta)(x+\gamma)(x+\delta)$
(c) $2(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$
(d) none of these

18. If α, β be the roots of the equation $ax^2 + bx + c = 0$, let $S_n = \alpha^n + \beta^n$ for $n \geq 1$. Then the value of

$$\Delta = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix} \text{ is}$$

(a) $\frac{(a+b+c)^2}{4}$ (b) $\frac{(b^2-4ac)}{a+b+c}$
(c) $\frac{(a+b+c)^2(b^2-4ac)}{a^4}$
(d) none of these.

19. If $f(x) = \begin{vmatrix} x & \cos x & e^{x^2} \\ \sin x & x^2 & \sin x \\ \tan x & 1 & 2 \end{vmatrix}$, then the value of $\int_{-\pi/2}^{\pi/2} f(x)$ is equal to

(a) 1 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) zero

20. The system of equations $x - y\cos\theta + z\cos 2\theta = 0$, $-x\cos\theta + y - z\cos\theta = 0$, $x\cos 2\theta - y\cos\theta + z = 0$, has non-trivial solution for θ equal to

(a) $\pi/3$ (b) $\pi/6$
(c) $2\pi/3$ (d) for all values of θ.

21. If $A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 3 & 2 \\ 3 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 & 4 \\ 1 & 3 & 2 \\ -1 & 1 & 5 \end{bmatrix}$, then $A + B$ is

(a) $\begin{bmatrix} 1 & -2 & 4 \\ 3 & 3 & 2 \\ 2 & 1 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 8 \\ 3 & 3 & 4 \\ 2 & 1 & 10 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -4 & 8 \\ 3 & 6 & 4 \\ 2 & 2 & 10 \end{bmatrix}$ (d) none of these.

22. The matrix $\begin{bmatrix} \lambda & 7 & -2 \\ 4 & 1 & 3 \\ 2 & -1 & 2 \end{bmatrix}$ is a singular matrix if λ is

(a) $-2/5$ (b) $5/2$
(c) -5 (d) none of these.

23. If A and B are skew symmetric matrices of order n, then

(a) $A + B$ is skew symmetric
(b) $A + B$ is symmetric
(c) $A + B$ is a diagonal matrix
(d) $A + B$ is a zero matrix.

24. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}$, then the value of $|\text{adj } A|$ is equal to

(a) 5 (b) 0
(c) 1 (d) none of these.

25. The rank of matrix $\begin{bmatrix} 4 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix}$ is

(a) 4 (b) 3
(c) 2 (d) none of these.

26. The system of equations

$x + y + z = 2$
$2x - y + 3z = 5$
$x - 2y - z + 1 = -1$

written in matrix form is

(a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & -2 & -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

(d) none of these.

27. If $A = \begin{bmatrix} 0 & f & g \\ -f & 0 & h \\ -g & -h & 0 \end{bmatrix}$, then A is

(a) diagonal (b) upper triangular
(c) symmetric (d) skew symmetric.

28. If A and B are two matrices such that $A + B$ and AB are both defined, then

(a) A and B can be any matrices.
(b) A, B are square matrices not necessarily of same order.
(c) A, B are square matrices of same order
(d) number of columns of A = number of rows of B.

29. The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the matrix of reflection in the line

(a) $x = 1$ (b) $y = 1$
(c) $x + y = 1$ (d) $x = y$.

30. If $\begin{bmatrix} x+1 & 5 \\ 4 & y-2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 4 & 3 \end{bmatrix}$, then

(a) $x = 2, y = 5$ (b) $x = 4, y = 2$
(c) $x = 2, y = 4$ (d) $x = 4, y = 5$.

31. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then $A^2 - 5A + 6I =$

(a) $\begin{bmatrix} +1 & -1 & -5 \\ -1 & -1 & 4 \\ -3 & -10 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$
(c) O (d) I

32. If $A = (a_{ij})$ is a scalar matrix of order $n \times n$ such that $a_{ij} = k$ for all i, then $|A|$ is equal to

(a) nk (b) $n + k$ (c) n^k (d) k^n

33. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, then A^n is equal to

(a) $\begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & n \\ 0 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 2n \\ 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

34. $\begin{bmatrix} 2+x & 3 & 4 \\ 1 & -1 & 2 \\ x & 1 & -5 \end{bmatrix}$ is a singular matrix, then x is

(a) 13/25 (b) –25/13 (c) 5/13 (d) 25/13

35. Let $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$G(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$,

then $[F(\alpha)\, G(\beta)]^{-1}$ is equal to

(a) $F(\alpha) - G(\beta)$ (b) $-F(\alpha) - G(\beta)$
(c) $[F(\alpha)]^{-1}\, [G(\beta)]^{-1}$ (d) $[G(\beta)]^{-1}\, [F(\alpha)]^{-1}$

36. For each real number x such that $-1 < x < 1$, let $A(x)$ be the matrix $(1-x)^{-1/2}\begin{pmatrix} 1 & -x \\ -x & 1 \end{pmatrix}$ and $z = \dfrac{x+y}{1+xy}$, then

(a) $A(z) = A(x) + A(y)$ (b) $A(z) = A(x)\, [A(y)]^{-1}$
(c) $A(z) = A(x)\, A(y) \div \sqrt{1+xy}$
(d) $A(z) = A(x) - A(y)$

37. If B is a non-singular matrix and A is a square matrix, then det $(B^{-1}AB)$ is equal to

(a) det(B) (b) det(A)
(c) det(B^{-1}) (d) det(A^{-1})

38. If $\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then

(a) $a = 1, b = 1$ (b) $a = \cos\theta, b = \sin\theta$
(c) $a = \sin 2\theta, b = \cos 2\theta$
(d) $a = \cos 2\theta, b = \sin 2\theta$

39. If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, then

(a) $\alpha = a^2 + b^2, \beta = ab$
(b) $\alpha = a^2 + b^2, \beta = 2ab$
(c) $\alpha = a^2 + b^2, \beta = a^2 - b^2$
(d) $\alpha = 2ab, \beta = a^2 + b^2$

40. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $(1, a, a^2)$, $(1, b, b^2)$, $(1, c, c^2)$ are noncoplanar, then abc =

(a) 2 (b) – 1
(c) 1 (d) 0

41. If the system of equations $x + 2ay + az = 0$, $x + 3by + bz = 0$, $x + 4cy + cz = 0$ has non-zero solution, then a, b, c are in

(a) A.P (b) G.P
(c) H.P (d) none of these

42. If $1, \omega, \omega^2$ are the cube roots of unity,

then $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix} =$

(a) 0 (b) 1
(c) ω (d) ω^2

43. If $A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, then

(a) A is zero matrix

(b) $A^2 = I$

(c) A^{-1} does not exist

(d) $A = (-1)\, I$

44. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$, then α =

(a) – 2 (b) 5 (c) 2 (d) – 1

45. If $A^2 - A + I = O$, then A^{-1} =

(a) A (b) $A + I$

(c) $I - A$ (d) $A - I$

46. If $a^2 + b^2 + c^2 = -2$ and

$$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix},$$

then $f(x)$ is a polynomial of degree

(a) 0 (b) 1 (c) 2 (d) 3

47. The system of equations

$\alpha x + y + z = \alpha - 1$

$x + \alpha y + z = \alpha - 1$

$x + y + \alpha z = \alpha - 1$

has no solution if α is

(a) – 2 or 1 (b) – 2

(c) 1 (d) – 1

48. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $a, b, \in N$, then

(a) there exists exactly one B such that $AB = BA$

(b) there exist infinitely many B's such that $AB = BA$

(c) there cannot exist any B such that $AB = BA$

(d) there exist more than 1 but finite number of B's such that $AB = BA$

49. If $A = \begin{bmatrix} -1 & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, then A^3 =

(a) $\frac{A}{4}$ (b) $\frac{A}{8}$ (c) $\frac{I}{4}$ (d) $\frac{I}{8}$

50. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then A^n =

(a) $2^{n-1} A - (n-1) I$

(b) $nA - (n-1) I$

(c) $2^{n-1} A + (n-1) I$

(d) $nA + (n-1) I$

51. If A and B are 3 × 3 matrices such that $A^2 - B^2 = (A - B)(A + B)$, then

(a) either A or B is zero matrix

(b) either A or B is unit matrix

(c) $A = B$

(d) $AB = BA$

52. If $a_1, a_2, a_3, \ldots$ are in $G.P.$, then

$$\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$$ is equal to

(a) 0 (b) 1 (c) 2 (d) 4

53. The homogeneous system of equations

$$\begin{bmatrix} 2 & a+b+c+d & ab+cd \\ a+b+c+d & 2(a+b)(c+d) & ab(c+d)+cd(a+b) \\ ab+cd & ab(c+d)+cd(a+b) & 2abcd \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

= 0 has non-trivial solutions only if

(a) $a + b + c + d = 0$

(b) $ab + cd = 0$

(c) $ab(c + d) + cd(a + b) = 0$

(d) for any a, b, c, d

54. If $x = \frac{a}{b-c}$, $y = \frac{b}{c-a}$, $z = \frac{c}{a-b}$, then

(a) $xy + yz + zx = 0$ (b) $xy + yz + zx = -1$

(c) $xy + yz + zx = 1$ (d) $x + y + z = 1$

55. If the equations $a(y + z) = x$, $b(z + x) = y$, $c(x + y) = z$ have non-trivial solutions, then $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}$ =

(a) 1 (b) 2 (c) – 1 (d) – 2

56. If the equations

$ax + 4y + z = 0$, $bx + 3y + z = 0$,

$cx + 2y + z = 0$ have non-trivial solution, then a, b, c are in

(a) A.P (b) G.P

(c) H.P (d) none of these

57. $\begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{(n+1)}P_{n+1} & {}^{(n+2)}P_{n+2} \\ {}^nC_n & {}^{(n+1)}C_{n+1} & {}^{(n+2)}C_{n+2} \end{vmatrix}$ =

(a) $n\,(n!)$ (b) $(n + 1)\,(n + 1)!$

(c) $(n + 2)(n + 2)!$ (d) $(n^2 + n + 1)\, n!$

58. If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ such that $|a_1| > |a_2| + |a_3|$, $|b_2| > |b_1| + |b_3|$, $|c_3| > |c_1| + |c_2|$, then A is

(a) orthogonal (b) singular
(c) non singular (d) non orthogonal.

59 . If $A = \begin{bmatrix} 23 & 1+i & -i \\ 1-i & -31 & 4-5i \\ i & 4+5i & 17 \end{bmatrix}$, then det A is

(a) complex number with positive real part
(b) complex number with negative imaginary part
(c) pure imaginary (d) real

60. If the matrix $A = \begin{bmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{bmatrix}$ is singular, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} =$

(a) 1 (b) – 1
(c) abc (d) $a + b + c$

61. The matrix $\begin{bmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{bmatrix}$ is singular if

(a) $a = 0$ (b) $a + b = 0$
(c) $a - b = 0$ (d) $a + b + c = 0$

62. If $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = k\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & a \end{vmatrix}$, then $k =$

(a) 0 (b) 1 (c) 2 (d) 3

63. $\begin{vmatrix} 2xy & x^2 & y^2 \\ x^2 & y^2 & 2xy \\ y^2 & 2xy & x^2 \end{vmatrix} =$

(a) $(x^3 + y^3)^2$ (b) $(x^2 + y^2)^3$
(c) $-(x^2 + y^2)^3$ (d) $-(x^3 + y^3)^2$

64. If x is a complex root of the equation $\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} + \begin{vmatrix} 1-x & 1 & 1 \\ 1 & 1-x & 1 \\ 1 & 1 & 1-x \end{vmatrix} = 0$,

then $x^{2007} + x^{-2007} =$

(a) 1 (b) – 1 (c) – 2 (d) 2

65. If $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ for $xy \neq 0$, then D is divisible by

(a) both x and y (b) x but not y
(c) y but not x (d) neither x nor y.

66. The value of $\begin{vmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & \cos 2\beta \\ \sin\alpha & \cos\alpha & \sin\beta \\ -\cos\alpha & \sin\alpha & \cos\beta \end{vmatrix}$ is independent of

(a) α (b) β
(c) α, β (d) none of these

67. Let $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$. If $\det(A^2) = 25$, then $|\alpha|$ is

(a) 1 (b) 1/5 (c) 5 (d) 5^2.

68. Let $a_n = \int_0^{\pi/2} \frac{1-\cos 2nx}{1-\cos 2x}\,dx$, then the value of $\begin{vmatrix} \frac{\pi}{2} & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$ equals

(a) 0 (b) 1
(c) 3 (d) None of these

69. If the value of determinant $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ is positive, then

(a) $abc > -11$ (b) $abc < -8$
(c) $abc > -8$ (d) None of these

70. If $D_k = \begin{vmatrix} 1 & 2k & 2k-1 \\ n & n^2+n+1 & n^2 \\ n & n^2+n & n^2+n+1 \end{vmatrix}$ & $\sum_{k=1}^{n} D_k = 72$, then value of n equals.

(a) 4 (b) 6 (c) 7 (d) 8

71. The value of m^{th} order determinant of a matrix A is 15, then the value of determinant formed by the cofactors of A will be

(a) $(15)^m$ (b) 15^{2m}
(c) $(15)^{m-1}$ (d) $(15)^{2m-1}$

72. If $A + B + C = \pi$ then $\begin{vmatrix} \tan(A+B+C) & \tan C & \cos C \\ \tan(A+B) & 0 & \sin B \\ \cos(A+B) & -\sin B & 0 \end{vmatrix}$ equals

(a) – 1 (b) 0
(c) 1 (d) None of these

73. If ω is cube root of unity and $x + y + z = a$, $x + \omega y + \omega^2 z = b$, $x + \omega^2 y + \omega z = c$ then which of the following is not correct?

(a) $x = \dfrac{a+b+c}{3}$ (b) $y = \dfrac{a+b\omega^2+\omega c}{3}$

(c) $z = \dfrac{a+b\omega+\omega^2 c}{3}$ (d) None of these

74. If $a \neq p, b \neq q, c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix}$ vanish, then the value of $\dfrac{p}{p-a} + \dfrac{q}{q-b} + \dfrac{r}{r-c}$ equals

(a) 2 (b) – 2

(c) 0 (d) None of these

75. The value of the determinant

$\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}$ equals

(a) $\dfrac{(x-y)(y-z)(z-x)}{6}$

(b) $\dfrac{xyz(x-y)(y-z)(z-x)}{12}$

(c) $\dfrac{(x-y)(y-z)(z-x)}{12}$

(d) $(xyz)(x-y)(y-z)(z-x)$

76. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 3 & -1 & 9 \end{pmatrix}$ then det (Adj (Adj A)) equals

(a) 11 (b) 121 (c) 1331 (d) 14641

77. If $A = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 4 & \log_y z \\ \log_z x & \log_z y & 6 \end{vmatrix}$ $\forall\, x, y, z > 0$, then what is the value of A?

(a) 15 (b) 0 (c) – 1 (d) 1

78. If $A = \begin{vmatrix} a+b & a+b+c & a+2+b+c \\ b+c & c+d+e & A+B+C+D \\ c+d & A+B+C & p+q+r+s \end{vmatrix}$, then number of decompositions of A are

(a) 12 (b) 6 (c) 8 (d) 24

79. Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, then A^n equals

(a) $\begin{bmatrix} 4^n & 0 & 0 \\ 0 & 4^{n-1} & 0 \\ 0 & 0 & 4^{n-2} \end{bmatrix}$

(b) $\begin{bmatrix} 2^{2n} & 0 & 0 \\ 0 & 2^{2n} & 0 \\ 0 & 0 & 2^{2n} \end{bmatrix}$

(c) $\begin{bmatrix} 4^{n-1} & 0 & 0 \\ 0 & 4^{n-1} & 0 \\ 0 & 0 & 4^{n-1} \end{bmatrix}$

(d) $\begin{bmatrix} |A|^n & 0 & 0 \\ 0 & |A|^n & 0 \\ 0 & 0 & |A|^n \end{bmatrix}$

80. One factor of the determinant $A = \begin{vmatrix} 1 & a & b & c \\ a & b & c & 1 \\ b & c & 1 & a \\ c & 1 & a & b \end{vmatrix}$ is

(a) $a + b + c$ (b) $(1 + b + c)$

(c) $1 + a + b + c$ (d) None of these

81. If $D = \text{diag}\,(d_1, d_2, d_3, ..., d_n)$ where $d_i \neq 0\ \forall\, i$, then D^{-1} equals

(a) D (b) diag $(d_1^n, d_2^n, ...\ d_n^n)$

(c) I_n

(d) diag $(d_1^{-1}, d_2^{-1}, d_3^{-1}, ...\ d_n^{-1})$

82. If $A = (a_{ij})_{m \times n}$ is a matrix of rank r then

(a) $r <$ Min (m, n)

(b) $r =$ Maximum of (m, n)

(c) $r =$ Minimum of (m, n)

(d) $r \leq$ Minimum of (m, n)

83. The rank of the matrix $A = \begin{pmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{pmatrix}_{2\times 4}$ is

(a) 4 (b) 3

(c) 2 (d) None of these

84. The characteristic roots of the matrix

$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$ is/are given by

(a) 1, – 4 (b) 1, – 4, 7

(c) – 1, 4, – 7 (d) None of these

85. If $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ then A^n equals.

(a) $\begin{pmatrix} 3n & -4n \\ n & -n \end{pmatrix}$ (b) $\begin{pmatrix} 3^n & 4^n(-1)^n \\ 1^n & (-1)^n \end{pmatrix}$

(c) $\begin{pmatrix} 3+n & -(4+n) \\ n & -n \end{pmatrix}$ (d) None of these

86. If matrix A is an circulant matrix whose elements of first row are $a, b, c,$ all > 0 such that $abc = 1$ and $A^TA = 1$, then $a^3 + b^3 + c^3$ equals

(a) 0 (b) 4 (c) 1 (d) 3

87. If $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$

then $\int_{-\pi}^{\pi} f(x)\,dx$ equals

(a) 0 (b) $-\pi$

(c) π (d) None of these

88. If $\begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix}^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

then the least value of k equals ($k \neq 0$)

(a) 1 (b) 2 (c) -1 (d) 3

89. If A_1, B_1, C_1 are respectively the co-factors of a_1, b_1, c_1 of the determinant

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then $\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix}$ equals

(a) $a_1 a_3 \Delta$ (b) $(a_1 - b_1)\Delta$

(c) $a_1\Delta$ (d) None of these

90. Let $\begin{vmatrix} x^2+x & 2x-1 & x+3 \\ 3x+1 & 2+x^2 & x^3-3 \\ x-3 & x^2+4 & 2x \end{vmatrix} = px^7 + qx^6 + rx^5 + sx^4 + tx^3 + ux^2 + vx + w$

then which of the following is not true

(a) $w = 21, v = 75$ (b) $q = 0, s = -4$

(c) $p = -1, t = -8$ (d) $p = q = -1$

91. The value of the determinant

$\begin{vmatrix} 1 & {}^mC_1 & {}^mC_2 \\ 1 & {}^{m+1}C_1 & {}^{m+1}C_2 \\ 1 & {}^{2+m}C_1 & {}^{m+2}C_2 \end{vmatrix}$ equals

(a) 0 (b) -1 (c) $m!$ (d) 1

92. If $f(x) = \begin{vmatrix} \sin x & 1 & 0 \\ 1 & 2\sin x & 1 \\ 0 & 1 & 2\sin x \end{vmatrix}$, then $\int_{-\pi/2}^{\pi/2} f(x)\,dx$

equals

(a) 0 (b) -1 (c) 1 (d) $\frac{3\pi}{2}$

93. If $f(x) = \begin{vmatrix} x & x^2 & 2x \\ \sin x & \tan x & \sin 2x \\ \cos x & -x^3 & 5x \end{vmatrix}$ then $\lim_{x \to 0} \frac{f'(x)}{x}$

equals

(a) 2 (b) 3 (c) -2 (d) -4

94. If $A = \begin{pmatrix} 0 & i-\sin x & i-\cos x \\ \sin x - i & 0 & \sin x - i \\ \cos x - i & -\sin x + i & 0 \end{pmatrix}$ then,

$|A|$ equals.

(a) 0 (b) $\sin x$ (c) $\cos x$ (d) 1

95. If A is invertible matrix and B is another matrix then

(a) rank (AB) = rank (A)

(b) rank (AB) = rank (B)

(c) rank (AB) > rank (A)

(d) rank (AB) > rank (B)

96. Suppose $p, q, r \neq 0$ and system of equations

$(p + a)x + by + cz = 0$

$ax + (q + b)y + cz = 0$

$ax + by + (r + c)z = 0$ has a non-trivial solution,

then value of $\frac{p}{a} + \frac{q}{b} + \frac{r}{c}$ is

(a) -1 (b) 0 (c) 1 (d) 2

97. If the system of the equations

$2x + 3y = -1$, $3x + y = 2$, $\lambda x + 2y = \mu$ is consistent, then

(a) $\lambda - \mu = 2$ (b) $\lambda + \mu = -1$

(c) $\lambda + \mu = 3$ (d) $\lambda - \mu + 8 = 0$

98. Let $A = \begin{pmatrix} 0 & \sin\alpha & \sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha\cos\beta \\ -\sin\alpha\sin\beta & -\cos\alpha\cos\beta & 0 \end{pmatrix}$,

then

(a) $|A|$ is independent of α and β

(b) A^{-1} depends only on α

(c) A^{-1} depends only on β

(d) none of these

99. If A is a square matrix of order 2×2 such that $A^2 = O$, then

(a) $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ where α, β, γ are numbers such that $\alpha^2 + \beta\gamma = 0$

(b) $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ with $\alpha = \pm\beta$

(c) $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ with $\alpha^2 + \beta^2 = 1$

(d) none of these

MCQs — More than One Correct Option

1. If A and B are two matrices such that $AB = A$ and $BA = B$, then B^2 is equal to

(a) B (b) A (c) BA (d) AB

2. If $A = [a_{ij}]$ is a square matrix of even order such that $a_{ij} = i^2 - j^2$, then

(a) A is a skew-symmetric matrix

(b) A is a symmetric matrix and $|A|$ is a square

(c) A is a symmetric matrix and $|A| = 0$

(d) $|A|$ is the square of some quantity

3. If A, B are two $n \times n$ non-singular matrices, then
(a) AB is non-singular
(b) AB is singular (c) $(AB)^{-1} = B^{-1}A^{-1}$
(d) $(AB)^{-1}$ may or may not exist

4. Let A be an invertible matrix. Which of the following are true?
(a) $(A^T)^{-1} = (A^{-1})^T$ (b) $A^{-1} = |A|^{-1}$
(c) $(A^2)^{-1} = (A^{-1})^2$ (d) $|A^{-1}| = |A|^{-1}$

5. If A and B are square matrices such that $B = -A^{-1}BA$, then
(a) $AB + BA = O$ (b) $(A + B)^2 = A^2 + B^2$
(c) $(A + B)^2 = A^2 + 2AB + B^2$
(d) $(A + B)^2 = A + B$

6. If A is a 3×3 matrix and B is its adjoint such that $|B| = 64$, then $|A|$ is equal to
(a) 64 (b) ±64 (c) 8 (d) –8

7. If $\Delta = \begin{vmatrix} x & 2y-z & -z \\ y & 2x-z & -z \\ y & 2y-z & -z \end{vmatrix}$, then
(a) $(x - y)$ is a factor of Δ
(b) $(x - y)^2$ is a factor of Δ
(c) $(x - y)^3$ is a factor of Δ
(d) Δ is independent of z

8. Let $\Delta = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$, then
(a) Δ is independent of θ
(b) Δ is independent of ϕ
(c) Δ is a constant (d) $\left[\dfrac{d\Delta}{d\theta}\right]_{\theta=\pi/2} = 0$

9. If $\Delta = \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a \\ a+2d & a & a+d \end{vmatrix}$, then
(a) Δ depends on a (b) Δ depends on d
(c) Δ is a constant (d) all of these.

10. Let $a, b > 0$ and $\Delta = \begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix}$, then
(a) $a + b - x$ is a factor of Δ
(b) $x^2 + (a + b)x + a^2 + b^2 - ab$ is a factor of Δ
(c) $\Delta = 0$ has two real roots, if $a = b$
(d) none of these.

11. Let $\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$, then
(a) $1 - x^3$ is a factor of Δ
(b) $(1 - x^3)^2$ is a factor of Δ
(c) $\Delta(x) = 0$ has 4 real roots
(d) $\Delta'(1) = 0$

12. Let $f(x) = \begin{vmatrix} 1/x & \log x & x^n \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix}$, then
(a) $f^n(1)$ is independent of a
(b) $f^n(1)$ is independent of n
(c) $f^n(1)$ depends on a and n
(d) $y = a(x - f^n(1))$ represents a straight line through the origin.

13. Suppose $f_1(x, y) = a_1x + a_2y$ and $f_2(x, y) = b_1x^2 + b_2xy + b_3y^2$ and
$$\Delta = \begin{vmatrix} f_2(x_1, y_1) & y_1f_1(x_1, y_1) & y_1^2 \\ f_2(x_2, y_2) & y_2f_1(x_2, y_2) & y_2^2 \\ f_2(x_3, y_3) & y_3f_1(x_3, y_3) & y_3^2 \end{vmatrix}$$
(a) Δ is independent of a_2, b_2 and b_3
(b) Δ depends on x_1, x_2, x_3 but not on y_1, y_2, y_3
(c) Δ depends on y_1, y_2, y_3 but not on x_1, x_2, x_3
(d) for $1 \le r, s \le 3$, $r > 0$, $s > 0$, $x_ry_s - x_sy_r$ is a factor of Δ

14. Suppose a_1, a_2, a_3 are in A.P. and b_1, b_2, b_3 are in H.P. and let $\Delta = \begin{vmatrix} a_1-b_1 & a_1-b_2 & a_1-b_3 \\ a_2-b_1 & a_2-b_2 & a_2-b_3 \\ a_3-b_1 & a_3-b_2 & a_3-b_3 \end{vmatrix}$, then
(a) Δ is independent of $a_1, a_2, a_3, b_1, b_2, b_3$
(b) $a_1 - \Delta$, $a_2 - 2\Delta$, $a_3 - 3\Delta$ are in A.P.
(c) $b_1 + \Delta$, $b_2 + \Delta^2$, $b_3 + \Delta$ are in H.P.
(d) none of these

15. Let $\Delta = \begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & a+x\sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & b+x\sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & c+x\sin\gamma \end{vmatrix}$, then
(a) Δ is independent of x
(b) $y = \Delta$ represents a straight line parallel to x-axis
(c) $f(x) = \Delta$ has a local maximum at $x = (\alpha + \beta + \gamma)/3$
(d) $f(x) = \Delta$ has a local minimum at $x = (\alpha + \beta + \gamma)/3$

16. The value of the determinant
$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & 1 \\ \sin\beta & \cos\beta & 1 \\ \sin\gamma & \cos\gamma & 1 \end{vmatrix}, \text{ equals}$$

(a) $-4\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}$
(b) $\sin\alpha + \sin\beta + \sin\gamma$
(c) $\sin(\beta-\gamma) + \sin(\gamma-\alpha) + \sin(\alpha-\beta)$
(d) none of these

17. The determinant

$$\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha\sin\beta \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha\sin\beta \\ 0 & -\sin\beta & \cos\beta \end{vmatrix} \text{ is}$$

(a) independent of α (b) independent of β
(c) $(\sin\alpha)x + (\cos\alpha)y = \Delta$ is a tangent line to the unit circle $x^2 + y^2 = 1$
(d) none of these.

18. Let $\Delta = \begin{vmatrix} 2\cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$, then

(a) $\int_0^{\pi} f(x)dx = 0$
(b) maximum value of $f(x)$ is 4
(c) $\lim_{x\to\pi/2} f(x) = 0$ (d) $f'(0) = 0$

19. If A, B and C are the angles of a triangle then

$$\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -2 \end{vmatrix} \text{ is dependent of}$$

(a) A (b) B
(c) C (d) none of these

20. Suppose x, y, z are positive and none of x, y, z is 1. If $\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \sin(x+y) & -\cos(x+y) & \sin^2 z \end{vmatrix}$ then Δ is independent of

(a) x (b) y
(c) z (d) none of these

21. The determinant

$$\Delta = \begin{vmatrix} x & -\sin\theta & -\cos\theta\sin\phi \\ \sin\theta & x & -\cos\theta\cos\phi \\ \cos\theta\sin\phi & \cos\theta\cos\phi & x \end{vmatrix}$$

is independent of

(a) x (b) θ
(c) ϕ (d) none of these

22. If D_1 and D_2 are two 3×3 diagonal matrices, then

(a) D_1D_2 is a diagonal matrix
(b) $D_1D_2 = D_2D_1$
(c) $D_1^2 + D_2^2$ is a diagonal matrix
(d) none of these

23. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then

(a) $A^{-n} = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix} \forall\ n \in N.$
(b) $\lim_{n\to\infty} \frac{1}{n} A^{-n} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$
(c) $\lim_{n\to\infty} \frac{1}{n^2} A^{-n} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
(d) none of these

24. Let A, B and C be 2×2 matrices with entries from the set of real numbers. Define* as follows : $A * B = 1/2\ (AB + BA)$, then

(a) $A * B = B * A$ (b) $A * A = A^2$
(c) $A * (B + C) = A * B + A * C$
(d) $A * I = A$

25. Let A and B are two matrices such that $AB = BA$, then for every $n \in N$.

(a) $A^nB = BA^n$ (b) $(AB)^n = A^nB^n$
(c) $(A + B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B + \ldots + {}^nC_nB^n$.
(d) $A^{2n} - B^{2n} = (A^n - B^n)(A^n + B^n)$

26. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then

(a) $A^2 - 4A - 5I_3 = 0$ (b) $A^{-1} = 1/5\ (A - 4I_3)$
(c) A^3 is invertible
(d) A^2 is invertible

27. Suppose $a_1, a_2, \ldots\ldots$ real numbers, with $a_1 \neq 0$. If $a_1, a_2, a_3, \ldots.$ are in A.P. then

(a) $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_5 & a_6 & a_7 \end{bmatrix}$ is singular
(b) the system of equations $a_1x + a_2y + a_3z = 0$, $a_4x + a_5y + a_6z = 0$, $a_7x + a_8y + a_9z = 0$ has infinite number of solutions
(c) $B = \begin{bmatrix} a_1 & ia_2 \\ ia_2 & a_1 \end{bmatrix}$ is non singular
(d) none of these.

28. If the equations $x + y = 1$, $(c + 2)\,x + (c + 4)\,y = 6$, $(c + 2)^2\,x + (c + 4)^2\,y = 36$ are consistent, then c =

(a) 1 (b) 2 (c) 3 (d) 4

29. The value of θ lying between $\theta = 0$ and $\theta = \frac{\pi}{2}$ and satisfying

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0 \text{ are}$$

(a) $\frac{7\pi}{24}$ (b) $\frac{5\pi}{24}$ (c) $\frac{11\pi}{24}$ (d) $\frac{\pi}{24}$

30. The value of θ for which the system of linear equations in x, y, z given as
$(\sin 3\theta)\, x - y + z = 0$
$(\cos 2\theta)\, x + 4y + 3z = 0$
$2x + 7y + 7z = 0$
has a non trivial solution

(a) $\frac{m\pi}{2}$ (b) $m\pi$

(c) $n\pi + (-1)^n \frac{\pi}{6}$ (d) None of these

Assertion Reason

Directions: In each of the following questions, a statement of Statement-1 is given and a corresponding statement of Statement-2 is given just below it. Of the statements, mark the correct answer as

(a) if both Statement-1 and Statement-2 are true and Statement-2 is the correct explanation of Statement-1
(b) if both Statement-1 and Statement-2 are true but Statement-2 is not the correct explanation of Statement-1
(c) if Statement-1 is true but Statement-2 is false
(d) if Statement-1 is false and Statement-2 is true.

1. **Statement-1 :** The product of two diagonal matrices of order 3×3 is also a diagonal matrix.

Statement-2 : Matrix multiplication is non-commutative.

2. **Statement-1 :** If a, b, c are distinct and x, y, z are not all zero given that $ax + by + cz = 0$, $bx + cy + az = 0$, $cx + ay + bz = 0$, then $a + b + c \neq 0$.

Statement-2 : $a^2 + b^2 + c^2 > ab + bc + ca$ if a, b, c are distinct.

3. **Statement-1 :** If $D(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

where f_1, f_2, f_3 are differentiable functions and $a_2, b_2, c_2, a_3, b_3, c_3$ are constants, then

$$\int D(x)\,dx = \begin{vmatrix} \int f_1(x)\,dx & \int f_2(x)\,dx & \int f_3(x)\,dx \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + c$$

Statement-2 : Integration of sum of several functions is equal to sum of integration of individual functions.

4. **Statement-1 :** There are only finitely many 2×2 matrices which commute with the matrix $\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$.

Statement-2 : If A is non-singular, then it commutes with I, adj A and A^{-1}.

5. **Statement-1 :** If A is skew symmetric of order 3 then its determinant should be zero.

Statement-2 : If A is square matrix, then $\det A = \det A' = \det(-A')$.

6. **Statement-1 :** The system of equations $x - ay + bz = 0$, $ax + y - cz = 0$, $-bx + cy + z = 0$ does not have non trivial solutions.

Statement-2 : If A is skew symmetric matrix, then $I - A$ is non singular.

7. Let A be skew-symmetric matrix, $B = (I - A)(I + A)^{-1}$ and let X and Y be column vectors comformable for multiplication with B.

Statement-1 : $(BX)^T (BY) = X^T Y$

Statement-2 : If A is skew-symmetric, then $(I + A)$ is non singular.

8. **Statement-1 :** If $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, then $A^3 + A^2 + A = I$

Statement-2 : If $\det(A - \lambda I) = C_0\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0$, then $C_0A^3 + C_1A^2 + C_2A + C_3I = 0$

9. **Statement-1 :** If the Matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, then $5A^{-1} = A^2 - 7A + 10I$

Statement-2 : If $\det(A - \lambda I) = \sum_{r=0}^{3} C_r \lambda^r$, then $C_0I + C_1A + C_2A^2 + C_3A^3 = 0$

10. **Statement-1 :** The equation $\begin{vmatrix} x & -2 & 1 \\ 2 & x & -3 \\ -1 & 3 & x \end{vmatrix} = 0$ has only one real root.

Statement-2 : If A is a real symmetric matrix, then $|A + xI| = 0$ has 3 real roots.

11. **Statement-1 :** The inverse of the matrix $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 9 & 8 & 7 \end{bmatrix}$ does not exist.

Statement-2 : The matrix $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 9 & 8 & 7 \end{bmatrix}$ is singular.

Passage Comprehension

PASSAGE 1 : If $abc = p$ and $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$, where A is, an orthogonal matrix. Then

1. The value of $a + b + c$ is
(a) 2 (b) p
(c) $2p$ (d) ± 1

2. The value of $ab + bc + ca$ is
(a) 0 (b) p
(c) $2p$ (d) $3p$

3. The value of $a^3 + b^3 + c^3$ is
(a) p (b) $2p$
(c) $3p$ (d) none of these

4. The equation whose roots are a, b, c is
(a) $x^3 - 2x^2 + p = 0$
(b) $x^3 - px^2 + px + p = 0$
(c) $x^3 - 2x^2 + 2px + p = 0$
(d) $x^3 \pm x^2 - p = 0$

PASSAGE 2 : Let A be $n \times n$ matrix with determinant $|A| \neq 0$, then

1. $|\text{adj } A| =$
(a) $|A|$ (b) $|A|^n$
(c) $|A|^{n-1}$ (d) $|A|^{n+1}$

2. $(\text{adj } A)^{-1} =$
(a) A (b) $\frac{A}{|A|}$
(c) $\frac{A}{|A|^n}$ (d) $\frac{A}{|A|^{n-1}}$

3. $|\text{adj adj } A| =$
(a) $|A|^{2n}$ (b) $|A|^{n^2}$
(c) $|A|^{(n-1)^2}$ (d) $|A|^{(n+1)^2}$

4. adj adj A =
(a) A (b) $|A|^n A$
(c) $|A|^{n-1} \cdot A$ (d) $|A|^{n-2} \cdot A$

PASSAGE 3 : Consider the matrix $A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$ and the homogeneous system $AX = \lambda X$, $X \neq O$

1. The number of distinct values of λ is
(a) 1 (b) 2
(c) 3 (d) 4

2. The H.C.F. of all the values of λ is
(a) 12 (b) 15
(c) 3 (d) 4

3. The product of all the values of λ is
(a) 140 (b) 144
(c) 152 (d) 162

4. The angle between two different column vectors X is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

PASSAGE 4 : Consider the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ then

1. $A^3 - 6A^2 + 10A - 5I =$
(a) O (b) $I + A$
(c) $A - I$ (d) $I - A$

2. $6A^{-1} =$
(a) $A^2 - 4A + 8I$ (b) $A^2 - 5A + 9I$
(c) $A^2 - 6A + 10I$ (d) $A^2 - 6A + 11I$

3. $A^4 =$
(a) $25A^2 - 66A + 42I$
(b) $25A^2 - 60A + 36I$
(c) $25A^2 + 66A - 42I$
(d) $25A^2 + 60A - 36I$

4. $A^{-2} =$
(a) $\frac{1}{36}[11A^2 - 60A + 85I]$
(b) $\frac{1}{18}[11A^2 - 30A + 25I]$
(c) $\frac{1}{12}[11A^2 - 60A - 85I]$
(d) $\frac{1}{36}[11A^2 + 60A - 85I]$

PASSAGE 5 : If $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, evaluate three roots $(\lambda_1, \lambda_2, \lambda_3)$ of λ from the equation $|A - \lambda I| = 0$. Construct a non-null invertible matrix X such that $AX = \lambda X$ where λ = diagonal $(\lambda_1, \lambda_2, \lambda_3)$.

1. The three roots of λ are
(a) 1, 0, 2 (b) –1, 1, 3
(c) 2, –2, 3 (d) 5, 1, –3

2. Trace of matrix A^n is

(a) $3^n + 1 + (-1)^n$ (b) $2^n + 1$

(c) $3^n - 2^n + 2$ (d) $3^n + 2^n - 2$

3. Matrix A satisfies

(a) $A^2 - (\lambda_1 + \lambda_2) A + \lambda_1\lambda_2 I = 0$

(b) $A^2 - (\lambda_1 + \lambda_3) A + \lambda_1\lambda_3 I = 0$

(c) $A^2 - (\lambda_2 + \lambda_3) A + \lambda_2\lambda_3 I = 0$

(d) none of these

PASSAGE 6 : A square matrix A is said to be invertible if $\exists$ a square matrix B such that

$AB = BA = I$

Here $B = A^{-1}$ or $A = B^{-1}$

Further a square matrix A is non-singular if $|A| \neq 0$ and it is singular if $|A| = 0$.

A is invertible $\Leftrightarrow$ A is non-singular

Again $A^{-1} = \dfrac{\text{adj. } A}{|A|}$

where adj. $A = \begin{bmatrix} A_{11} & A_{21} & A_{31}\ldots & A_{n1} \\ A_{12} & A_{22} & A_{32}\ldots & A_{n2} \\ A_{13} & A_{23} & A_{33}\ldots & A_{n3} \\ \ldots & \ldots & \ldots & \ldots \\ A_{1n} & A_{2n} & A_{3n}\ldots & A_{nn} \end{bmatrix}$

[A_{ij} = co-factor of a_{ij}]

$(AB)^{-1} = B^{-1} A^{-1}$ and $(A')^{-1} = (A^{-1})'$

1. If $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, then $(A^{-1})^3$ is equal to

(a) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$ (b) $\frac{1}{27}\begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$

(c) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & -27 \end{bmatrix}$ (d) $\frac{1}{27}\begin{bmatrix} -1 & -26 \\ 0 & -27 \end{bmatrix}$

2. If A is square matrix such that $A^2 = I$, then A^{-1} is equal to

(a) $A + I$ (b) A

(c) 0 (d) $2A$

3. If D = diagonal $(d_1, d_2, d_3, \ldots, d_n)$, where $d_i \neq 0$ for all $i = 1, 2, \ldots, n$, then D^{-1} is equal to

(a) D (b) diag. $(d_1^{-1}, d_2^{-1}, \ldots d_n^{-1})$

(c) I_n (d) none of these

PASSAGE 7 : Let A be any n-rowed square matrix. Then a square matrix B such that $AB = BA = I$, is called the inverse of A and is denoted by A^{-1}. We can find A^{-1} by the formula

$$A^{-1} = \frac{\text{Adj. } A}{|A|} \qquad [|A| \neq 0]$$

Again $AA^{-1} = A^{-1}A = I$

Again $(AB)^{-1} = B^{-1}A^{-1}$

Also $|AB| = |A| \cdot |B|$

1. If A is an invertible matrix, then $\det(A^{-1})$ is equal to

(a) $\det A$ (b) $\dfrac{1}{\det A}$

(c) 1 (d) none of these

2. From the matrix equation $AB = AC$, we can conclude $B = C$ provided

(a) A is singular (b) A is non-singular

(c) A is symmetric (d) none of these.

3. If $A^2 - A + I = 0$, then inverse of A is

(a) $A + I$ (b) A

(c) $A - I$ (d) $I - A$

PASSAGE 8 : Consider a system of linear equations in three variables x, y, z :

$$\begin{bmatrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{bmatrix} \qquad \ldots\ldots(i)$$

In symbolic form, we can write these as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

i.e., $AX = B$

If A is non-singular matrix, then $X = A^{-1}B$ gives the unique solution of (i). If A is singular i.e., $|A| = 0$, then the system will have no unique solution if (Adj A)$B = 0$ and the system has no solution *i.e.*, it is inconsistent if (Adj A) $B \neq 0$

where Adj $A = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$ and A_i = co-factor of a_i in A etc.

1. The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + Kz = 4$ has a unique solution if

(a) $K \neq 0$ (b) $-1 < K < 1$

(c) $-2 < K < 2$ (d) $K = 0$

2. The system of equations $x + 2y + 3z = 4$, $2x + 3y + 4z = 5$, $3x + 4y + 5z = 6$ has

(a) many solutions (b) no solution

(c) unique solution (d) none of these

3. If the system of equations $x + 2y - 3z = 1$, $(p + 2)z = 3$, $(2p + 1)y + z = 2$ is inconsistent,

then the value of p is

(a) –2 (b) 1

(c) 0 (d) 2

PASSAGE 9 :

Homogeneous Equations :

The system of equations $AX = B$ is said to be homogeneous if the constants d_1, d_2, d_3 are all zero *i.e.*, if $B = O$

i.e., $AX = O$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(i) If $|A| \neq 0$, then its only solution $X = O$, is called the trivial solution *i.e.*, $x = 0, y = 0, z = 0$ is the trivial solution.

(ii) If $|A| = 0$, then $AX = O$ has a non-trivial solution. It will have infinitely solutions.

1. If $x = cy + bz$, $y = bz + cx$, $z = bx + ay$ where x, y, z are not all zero, then

(a) $a^2 + b^2 + c^2 + 2abc = 0$

(b) $a^2 + b^2 + c^2 - 2abc = 1$

(c) $a^2 + b^2 + c^2 + 2abc = 1$

(d) none of these

2. Let λ and α be real. The set of all values of λ for which the system of linear equations

$\lambda x + (\sin \alpha)y + (\cos \alpha)z = 0$

$x + (\cos \alpha)y + (\sin \alpha)z = 0$

$-x + (\sin \alpha)y - (\cos \alpha)z = 0$

has a non-trival solution, is

(a) $[0, \sqrt{2}]$ (b) $[-\sqrt{2}, 0]$

(c) $[-\sqrt{2}, \sqrt{2}]$ (d) none of these

3. If the three linear equations $x + 4ay + az = 0$, $x + 3by + bz = 0$, $x + 2cy + cz = 0$, have a non-trivial solution, then a, b, c are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

PASSAGE 10 : Let A be any n-rowed square matrix. Then a square matrix B such that $AB = BA = I$, is called the inverse of A and is denoted by A^{-1}. We can find A^{-1} by the formula

$$A^{-1} = \frac{\text{Adj } A}{|A|} \qquad [|A| \neq 0]$$

Again $AA^{-1} = A^{-1}A = I$

Again $(AB)^{-1} = B^{-1}A^{-1}$

Also $|AB| = |A| \cdot |B|$

1. If A is an invertible matrix, then $\det(A^{-1})$ is equal to

(a) det A (b)

(c) 1 (d) none of these

2. From the matrix equation $AB = AC$, we can conclude $B = C$ provided

(a) A is singular (b) A is non-singular

(c) A is symmetric (d) none of these.

3. If $A^2 - A + I = 0$, then inverse of A is

(a) $A + I$ (b) A

(c) $A - I$ (d) $I - A$

Matrix — Match Type

1.

	Column I	Column II
(A)	If A satisfies the equation $x^3 - 5x^2 + 4x + \lambda = 0$, then A^{-1} exists, if λ is not equal to	(p) 3
(B)	If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then $\lim_{n\to\infty} \frac{1}{n} A^n = \begin{bmatrix} 0 & a\lambda \\ 0 & 0 \end{bmatrix}$. Then λ is equal to	(q) 0
(C)	For what value of λ, the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$, and $x + 2y + \lambda z = 12$ is inconsistent?	(r) 5
		(s) 1

2.

	Column I	Column II
(A)	If $A^2 = 8A + kI$, where $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, then k is	(p) 1
(B)	The matrix $\begin{bmatrix} \lambda & 7 & -2 \\ 4 & 1 & 3 \\ 2 & -1 & 2 \end{bmatrix}$ is a singular matrix, if λ is	(q) –7
(C)	If $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}$, then the value of $|\text{adj } A|$ is equal to	(r) 3
(D)	The rank of the matrix $\begin{bmatrix} -5 & 3 & 2 \\ 3 & 2 & -5 \\ 4 & -1 & -3 \end{bmatrix}$, is	(s) 2/5

(E) The rank of the matrix $\begin{bmatrix} 4 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix}$, is — (t) 2

3. A is a 3 × 3 matrix with determinant $|A| \neq 0$.

Column I	Column II
(A) $\lvert \text{adj } A \rvert$	(p) $\lvert A \rvert^2$
(B) $(\text{adj } A)^{-1}$	(q) $\lvert A \rvert^4$
(C) $\lvert \text{adj adj } A \rvert$	(r) $\dfrac{A}{\lvert A \rvert}$
(D) $(\text{adj adj } A)^{-1}$	(s) $\dfrac{A^{-1}}{\lvert A \rvert}$

4. The equations $x + \lambda y = 1$, $\lambda x + y = \lambda^2$ have

Column I	Column II
(A) unique solution if λ =	(p) 0
(B) no solution if λ =	(q) – 1
(C) infinity of solutions if λ =	(r) 1
	(s) 2

5. Let $f(x) = x(x - 1)$ and a be a real number.

Column I	Column II
(A) $\Delta_1 = \begin{vmatrix} f(0) & f(1) & f(2) \\ f(1) & f(2) & f(3) \\ f(2) & f(3) & f(4) \end{vmatrix}$	(p) –2!
(B) $\Delta_2 = \dfrac{1}{4}\begin{vmatrix} f(a) & f'(a) & f''(a) \\ f'(a) & f''(a) & f'''(a) \\ f''(a) & f'''(a) & f^{iv}(a) \end{vmatrix}$	(q) $-3! - 2!$
	(r) $2a^2 + 3a$
	(s) $2a^3 + 3a^2$

6. Let p be positive integer,

Column I	Column II
(A) $\Delta_1 = \begin{vmatrix} ^{p+2}C_2 & ^{p+3}C_2 & ^{p+4}C_2 \\ ^{p+3}C_2 & ^{p+4}C_2 & ^{p+5}C_2 \\ ^{p+4}C_2 & ^{p+5}C_2 & ^{p+6}C_2 \end{vmatrix}$	(p) $p^2 + p + 2$
(B) $\Delta_2 = \begin{vmatrix} 1 & ^{p}C_1 & ^{p}C_2 \\ 1 & ^{p+1}C_1 & ^{p+1}C_2 \\ 1 & ^{p+2}C_1 & ^{p+2}C_2 \end{vmatrix}$	(q) $p - 1$
	(r) -1
	(s) 1

7. Suppose $f(x)$ is a function satisfying the following conditions :

(a) $f(0) = 2$, $f(1) = 1$

(b) $f(x)$ has a maximum value at $x = 5/2$ and

(c) for all x,

$$f'(x) = \begin{vmatrix} 2ax & 2ax - 1 & 2ax + b + 1 \\ b & b + 1 & -1 \\ 2(ax + b) & 2ax + 2b + 1 & 2ax + b \end{vmatrix}$$

where a, b are constant. Then

Column I	Column II
(A) a	(p) –1/4
(B) b	(q) 1/4
	(r) –5/4
	(s) 7/4

8. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & \alpha \\ 4 & 7 & \beta \end{bmatrix}$.

Column I	Column II
(A) A is non-singular	(p) $\alpha = 6$
(B) A is singular	(q) $\alpha \neq 6$
	(r) $\beta = 12$
	(s) $\beta \neq 12$

9. The system of equations $x + y + z = 3$, $x + 2y + 3z = 6$, $x + 3y + \lambda z = \mu$ has

Column I	Column II
(A) no solution	(p) $\lambda = 5, \mu = 9$
(B) unique solution	(q) $\lambda = 5, \mu \neq 9$
(C) Infinite number of solutions	(r) $\lambda \neq 5, \mu \neq 9$
	(s) $\lambda \neq 5$

10. Let $A(\alpha, \beta) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & e^{\beta} \end{bmatrix}$, then

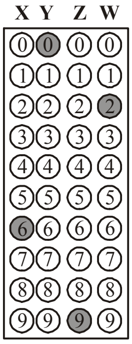
Column I	Column II
(A) $[A(\alpha, \beta)]^{-1}$	(p) $A(-\alpha, -\beta)$
(B) $[A(\alpha, \beta)]'$	(q) $A(-\alpha, \beta)$
	(r) $A(\alpha, \beta)$
	(s) $A(\alpha, -\beta)$

11.

Column I	Column II
(A) A is an orthogonal matrix	(p) $(14)^4$
(B) Determinant of a skew-symmetric matrix of even order is	(q) symmetric
(C) If $A = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix}$, then det. (adj(adj A)) is	(r) a perfect square
(D) For every square matrix A, $AA' - A'A$ is	(s) $\lvert A \rvert = \pm 1$

Integer Answer Type

In this section the answer to each of the questions is a single-digit integer, ranging from 0 to 9. If the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2, respectively, then the correct darkening of bubbles will look like the following.



1. If $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = k(xyz)$, then k is equal to

2. The value of λ for which the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$ have no solution is

3. If $A = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is symmetric, then $x =$

4. If the matrix $\begin{bmatrix} 1 & 3 & \lambda+2 \\ 2 & 4 & 8 \\ 3 & 5 & 10 \end{bmatrix}$ is singular, then λ is

5. The highest power of x in the expansion of the determinant $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ is

6. The system of linear equations $x + y - z = 6$, $x + 2y - 3z = 14$ and $2x + 5y - \lambda z = 9$ $(\lambda \in R)$ has a unique solution if λ is not equal to

7. If A, B, C are angles of a triangle ABC, then value of $\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix}$ is

8. If A is a square matrix of order 3 such that $A(\text{adj}A) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then $|\text{adj}A|$ is

9. The value of $\Delta = \begin{vmatrix} 0 & i-1000 & i-500 \\ 1000-i & 0 & 1000-i \\ 500-i & i-1000 & 0 \end{vmatrix}$ is equal to

10. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $A^2 + 5A$ is equal to nA, where n is

11. If A is a square matrix of order n such that $|\text{adj}\,(\text{adj}\,A)| = |A|^9$ then the value of n can be

12. The system of equations are

$$\frac{1}{x} + \frac{2}{y} + \frac{1}{z} = 1,\ \frac{3}{x} - \frac{4}{y} - \frac{2}{z} = 2,\ \frac{2}{x} + \frac{5}{y} - \frac{2}{z} = 3$$

then find the value of y.

13. If $\Delta_r = \begin{vmatrix} x & y & z \\ 2^r & 2 \times 3^r & 3 \times 4^r \\ 2(2^n - 1) & 3(3^n - 1) & 4(4^n - 1) \end{vmatrix}$, then find the value of $\sum_{r=1}^{n} \Delta_r$.

14. The value of the determinant

$$\Delta = \begin{vmatrix} 1! & 2! & 3! \\ 2! & 3! & 4! \\ 3! & 4! & 5! \end{vmatrix}$$ is $k!$, where k is equal to

15. The value of the determinant

$$\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 & (a_1-b_4)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 & (a_2-b_4)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 & (a_3-b_4)^2 \\ (a_4-b_1)^2 & (a_4-b_2)^2 & (a_4-b_3)^2 & (a_4-b_4)^2 \end{vmatrix}$$ is

16. Let three matrices $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$, then

$$tr(A) + tr\left(\frac{A(BC)}{2}\right) + tr\left(\frac{A(BC)^2}{4}\right) + tr\left(\frac{A(BC)^3}{B}\right) + \ldots + \infty$$

is equal to

17. The matrix $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is nilpotent of index

18. For how many values of x in the closed interval $[-4, -1]$, the matrix $\begin{bmatrix} 3 & -1+x & 2 \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{bmatrix}$ is singular?

19. If α, β and γ are such that $\alpha + \beta + \gamma = 0$, then $\begin{bmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{bmatrix}$ is equal to

20. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then the value of α^2 is equal to

QUESTIONS FROM PREVIOUS YEARS IIT-JEE/JEE Advanced

MCQs : One Correct Option

1. The no. of distinct real roots of

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0 \text{ in the interval } -\frac{\pi}{4} \le x \le \frac{\pi}{4}$$

is

(a) 0 (b) 2 (c) 1 (d) 3

(2001)

2. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, then value of α for which $A^2 = B$, is

(a) 1 (b) –1
(c) 4 (d) no real values

(2003)

3. If the system of equations $x + ay = 0$, $az + y = 0$ and $ax + z = 0$ has infinite solutions, then the value of a is

(a) –1 (b) 1
(c) 0 (d) no real values

(2003)

4. Given $2x - y + 2z = 2$, $x - 2y + z = -4$, $x + y + \lambda z = 4$ then the value of λ such that the given system of equations has no solution, is

(a) 3 (b) 1 (c) 0 (d) –3

(2004)

5. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$ then the value of α is

(a) ±1 (b) ±2
(c) ±3 (d) ±5 **(2004)**

6. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$A^{-1} = \left[\frac{1}{6}(A^2 + cA + dI)\right]$, then (c, d) is

(a) (–11, 6) (b) (–6, 11)
(c) (6, 11) (d) (11, 6) **(2005)**

7. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and

$Q = PAP^T$, then $P^T(Q^{2005})P$ is equal to

(a) $\begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ 0 & 2005 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{\sqrt{3}}{2} & 2005 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2005 \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}$

(2005)

8. The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is

(a) 0 (b) $2^9 - 1$
(c) 168 (d) 2 **(2010)**

MCQs : More than One Correct Option

9. Let X and Y be two arbitrary, 3×3 non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric?

(a) $Y^3Z^4 - Z^4Y^3$ (b) $X^{44} + Y^{44}$
(c) $X^4Z^3 - Z^3X^4$ (d) $X^{23} + Y^{23}$

(JEE Advanced 2015)

10. Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha?$$

(a) –4 (b) 9 (c) –9 (d) 4

(JEE Advanced 2015)

Assertion and Reason

Directions: In the following question, a statement of Statement-1 is given and a corresponding statement of Statement-2 is given just below it. Of the statements, mark the correct answer as

(a) if both Statement-1 and Statement-2 are true and Statement-2 is the correct explanation of Statement-1
(b) if both Statement-1 and Statement-2 are true but Statement-2 is not the correct explanation of Statement-1
(c) if Statement-1 is true but Statement-2 is false
(d) if Statement-1 is false and Statement-2 is true.

11. Consider the system of equations

$x - 2y + 3z = -1$

$-x + y - 2z = k$

$x - 3y + 4z = 1.$

Statement-1 : The system of equations has no solution for $k \neq 3$.

Statement-2 : The determinant

$$\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0, \text{ for } k \neq 3.$$

(2008)

Passage Comprehension

Comprehension-1

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, if U_1, U_2 and U_3 are columns matrices satisfying

$$AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \text{ and } AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

and U is 3×3 matrix whose columns are U_1, U_2, U_3 then answer the following questions.

12. The value of $|U|$ is

(a) 3 (b) –3

(c) 3/2 (d) 2 **(2006)**

13. The sum of the elements of U^{-1} is

(a) –1 (b) 0

(c) 1 (d) 3. **(2006)**

14. The value of $\begin{bmatrix} 3 & 2 & 0 \end{bmatrix} U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is

(a) 5 (b) 5/2

(c) 4 (d) 3/2. **(2006)**

Comprehension-2

Let A be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.

15. The number of matrices in A is

(a) 12 (b) 6

(c) 9 (d) 3 **(2009)**

16. The number of matrices A in A for which the system of linear equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, is

(a) less than 4

(b) at least 4 but less than 7

(c) at least 7 but less than 10

(d) at least 10 **(2009)**

17. The number of matrices A in A for which the system of linear equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is inconsistent, is

(a) 0 (b) more than 2

(c) 2 (d) 1

(2009)

Comprehension-3

Let p be an odd prime number and T_p be the following set of 2×2 matrices :

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, ..., p-1\} \right\}$$

18. The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det(A) divisible by p is

(a) $(p - 1)^2$ (b) $2(p - 1)$

(c) $(p - 1)^2 + 1$ (d) $2p - 1$ **(2010)**

19. The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p is

[Note : The trace of a matrix is the sum of its diagonal entries]

(a) $(p - 1)(p^2 - p + 1)$ (b) $p^3 - (p - 1)^2$

(c) $(p - 1)^2$ (d) $(p - 1)(p^2 - 2)$

(2010)

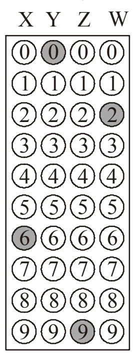
20. The number of A in T_p such that det (A) is not divisible by p is

(a) $2p^2$ (b) $p^3 - 5p$

(c) $p^3 - 3p$ (d) $p^3 - p^2$ **(2010)**

Integer Answer Type

In this section answer to each of the questions is a single digit integer, ranging from 0 to 9. If the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2 respectively, then the correct darkening of bubbles will look like the following



21. The number of all possible values of q, where $0 < \theta < \pi$, for which the system of equations
$(y + z)\cos^3\theta = (xyz)\sin 3\theta$;
$$x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$$
$(xyz)\sin 3\theta = (y + 2z)\cos 3\theta + y\sin 3\theta$ have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is _______ **(2010)**

22. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to _______ **(2010)**

23. Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix}$$ and

$$B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2\sqrt{k} & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If $\det(\text{adj } A) + \det(\text{adj } B) = 10^6$, then $[k]$ is equal to

[Note : adj M denotes the adjoint of a square matrix M and $[k]$ denotes the largest integer less than or equal to k].

(2010)

HINTS & SOLUTIONS

MCQs : One Correct Option

1. **(a) :** We can solve this problem by simply expanding the given determinant and equating it to zero.

Given determinant is $\begin{vmatrix} a+x & a & x \\ a-x & a & x \\ a-x & a & -x \end{vmatrix} = 0$

$\Rightarrow$ $(a + x)(-ax - ax) - a(-ax + x^2 - ax + x^2) + x(a^2 - ax - a^2 + ax) = 0$

$\Rightarrow$ $-2a^2x - 2ax^2 + 2a^2x - 2ax^2 = 0$

$\Rightarrow$ $-4ax^2 = 0 \Rightarrow x = 0.$

2. **(b) :** We have, $\Delta = \begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} R \rightleftarrows C$

$= \begin{vmatrix} 0 & q-p & r-p \\ p-q & 0 & r-q \\ p-r & q-r & 0 \end{vmatrix}$

Taking (–1) common from all the rows

$\Rightarrow$ $\Delta = (-1)^3 \Delta$ $\therefore \Delta = -\Delta \Rightarrow \Delta = 0.$

3. **(d) :** We will try to make the maximum number of zeros possible in any of the row or column to minimize the expansion.

Let, $\Delta = \begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix}$

Now making, $R_1 \to R_1 + R_2, R_2 \to R_2 - iR_3$

$\therefore$ $\Delta = \begin{vmatrix} 4+6i & 0 & 0 \\ 4-20i & 0 & 0 \\ 20 & 3 & i \end{vmatrix} = 0$

$\therefore$ $\Delta = 0 + 0.i = x + iy$ (given)

i.e. $x = 0, y = 0$

4. **(c) :** We will try to convert the given determinant in the simpler form and using the fact that if *abc* be a three digit number, then it can be written as $100a + 10b + c$.

Given determinant is,

$\Delta = \begin{vmatrix} 5 & 4 & 3 \\ 100x+50+1 & 100y+40+1 & 100z+30+1 \\ x & y & z \end{vmatrix}$

$(R_2 \to R_2 - 100R_3 - 10R_1)$

$\Delta = \begin{vmatrix} 5 & 4 & 3 \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = 5(z-y) - 4(z-x) + 3(y-x)$

$= x - 2y + z = 0$ ($\because x, y, z$ are in A.P.)

5. **(b) :** As the determinant has to vanish for all *a*, *b*, we have at least two rows or column identical.

After filling the first column in 2 × 2 × 2 ways and filling another column likewise, the remaining column can be filled in 2 × 2 × 2 ways, so, the number of ways = $2^3 \times 2^3 = 2^6$.

But each of the two ways give the same determinant.

The required number of determinants

$= 2^5 = 32.$

6. **(b) :** To find out the non-integral root of the given determinant first we will try to convert it into the simpler form and get a quadratic equation.

We have, $\begin{vmatrix} x & 2 & 5 \\ 3 & x & 3 \\ 5 & 4 & x \end{vmatrix} = 0$

$C_1 \to C_1 - C_3$

We get, $\begin{vmatrix} x-5 & 2 & 5 \\ 0 & x & 3 \\ 5-x & 4 & x \end{vmatrix} = 0$

$\Rightarrow$ $(x-5)\begin{vmatrix} 1 & 2 & 5 \\ 0 & x & 3 \\ -1 & 4 & x \end{vmatrix} = 0$

Now replacing $R_1 \to R_1 + R_3$

Then $(x-5)\begin{vmatrix} 0 & 6 & 5+x \\ 0 & x & 3 \\ -1 & 4 & x \end{vmatrix} = 0$

or $(x - 5)\{18 - x(5 + x)\} = 0$

$\therefore$ the non-integral roots are, the roots of $x^2 + 5x - 18 = 0$

$\alpha = \dfrac{-5+\sqrt{25+72}}{2} = \dfrac{-5+\sqrt{99}}{2}$

$\beta = \dfrac{-5-\sqrt{99}}{2}$

Sum of the roots = $\alpha + \beta$

$= -\dfrac{5}{2} + \dfrac{\sqrt{99}}{2} - \dfrac{5}{2} - \dfrac{\sqrt{99}}{2} = -5$.

7. **(c):** The given determinant is

$\Delta = \begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$

On expanding this determinant in powers of x, the constant term in the expansion = value of the determinant when $x = 0$

$$\Delta_{min} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

Replacing $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$

$$\begin{vmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = -1$$

8. **(c):** Each of the three options contain $m = 3$. When $m = 3$, the last two equations becomes $x + 2y + 3z = 10$ and $x + 2y + 3z = n$

Obviously, when $n = 10$ these equations become the same. So, we are left with only two independent equations to find the values of three unknowns.

Consequently, there will be infinite solutions when $m = 3$, $n = 10$

9. **(b):** The given system of equations is

$2x + 3y - 8 = 0$

$7x - 5y + 3 = 0$

$4x - 6y + \lambda = 0$

Equations are solvable when they are consistent.

For consistency we have,

$$\begin{vmatrix} 2 & 3 & -8 \\ 7 & -5 & 3 \\ 4 & -6 & \lambda \end{vmatrix} = 0$$

$\Rightarrow\ 2(-5\lambda + 18) - 3(7\lambda - 12) - 8(-42 + 20) = 0$

$\Rightarrow\ -31\lambda + 248 = 0$

$\Rightarrow\ \lambda = 8.$

10. **(a):** Expanding the given determinant.

$$\Delta = \cos\frac{\theta}{2}\left\{-\cos\frac{\theta}{2} + \cos\frac{\theta}{2}\right\} - 1\left\{-1 - \cos^2\frac{\theta}{2}\right\} + 1\left\{1 + \cos^2\frac{\theta}{2}\right\}$$

$$= 0 + 1 + \cos^2\frac{\theta}{2} + 1 + \cos^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} + 2$$

where maximum = 2 + 2(1) = 4

and whose minimum = 2 + 2 × 0 = 2

Hence Δ lies in the interval [2, 4].

11. **(b) :** It is given that, $z = \begin{bmatrix} 2+5i & 2-3i & 4+i \\ 8i & 3i & -5i \\ 2-5i & 2+3i & 4-i \end{bmatrix}$

To clearify where z is real, imaginary or equal to zero.

We know that if $z = x + iy$ is an imaginary number then its conjugate $\bar{z} = x - iy$

The complex conjugate of the given number $\bar{z}$ is

$$\bar{z} = \begin{vmatrix} 2-5i & 2+3i & 4-i \\ -8i & -3i & 5i \\ 2+5i & 2-3i & 4+i \end{vmatrix}$$

$$= -\begin{vmatrix} 2-5i & 2+3i & 4-i \\ 8i & 3i & -5i \\ 2+5i & 2-3i & 4+i \end{vmatrix}$$

Replacing, $R_1 \leftrightarrow R_3$

$$\therefore\ \bar{z} = \begin{vmatrix} 2+5i & 2-3i & 4-i \\ 8i & 3i & -5i \\ 2-5i & 2+3i & 4-i \end{vmatrix} = z$$

$\because\ \bar{z} = z \quad \Rightarrow z$ is purely real

12. **(d) :** It is given that

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0$$

We have to find out the type of the ΔABC.

To solve this problem first we try to convert the given determinant into the simpler form, then expand it.

The given determinant is

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0$$

taking $R_2 \to R_2 - R_3$

$$\Rightarrow\ 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0$$

using $R_3 \to R_3 - R_1$

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1-2a & 1-2b & 1-2c \end{vmatrix} = 0$$

Taking $R_3 \to R_3 + 2R_2 \ \Rightarrow \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$

Taking, $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Rightarrow \begin{vmatrix} a^2 & b^2 - a^2 & c^2 - a^2 \\ a & b - a & c - a \\ 1 & 0 & 0 \end{vmatrix}$$

$\Rightarrow\ (b^2 - a^2)(c - a) - (b - a)(c^2 - a^2) = 0$

$\Rightarrow\ (b - a)(c - a)(b - c) = 0$

$\Rightarrow\ a = b$ or $b = c$ or $c = a$

i.e. ΔABC is an isosceles triangle.

13. **(a) :** We have

$$f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & x(x^2-1) \end{vmatrix}, \ ;$$

$x \in R$

We have to find out the value of $f(27)$.

We will try to make the maximum number of zero possible in any of the row or column to minimize the expansion.

$$x \in R, f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & x(x^2-1) \end{vmatrix}$$

$$= x(x+1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x(x-1) & (x-1)(x-2) & x(x-1) \end{vmatrix}$$

$$= x(x+1)(x-1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix}$$

Taking $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$f(x) = x(x^2-1)\begin{vmatrix} 1 & 0 & 0 \\ 2x & -1-x & -x \\ 3x & -2-2x & -2x \end{vmatrix}$$

$$= x(x^2-1)\,x(x+1)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 1 & 1 \\ 3x & 2 & 2 \end{vmatrix}$$

$\because\ C_2 = C_3 \ \therefore\ f(x) = 0 \Rightarrow f(27) = 0.$

14. **(b) :** It is given that system of equations

$-ax + y + z = 0$ ...(1)

$x - by + z = 0$ ...(2)

and $x + y - cz = 0$ ...(3) $(a, b, c \neq 1)$

has a non-zero solution.

To find out the value of $\dfrac{1}{1+a}+\dfrac{1}{1+b}+\dfrac{1}{1+c}$.

For the homogeneous system of equations, the non-zero solution will exist only if $\Delta = 0$.

The equations (1), (2) & (3) are consistent in

x, y, z only if $\Delta = 0$ i.e. $\begin{vmatrix} -a & 1 & 1 \\ 1 & -b & 1 \\ 1 & 1 & -c \end{vmatrix} = 0$

$-a(bc-1) - 1(-c-1) + 1(1+b) = 0$

$abc = a + b + c + 2$ ...(A)

Also, $\dfrac{1}{1+a}+\dfrac{1}{1+b}+\dfrac{1}{1+c}$

$$= \frac{(1+b)(1+c)+(1+a)(1+c)+(1+a)(1+b)}{(1+a)(1+b)(1+c)}$$

$$= \frac{3+2(a+b+c)+(bc+ca+ab)}{1+(a+b+c)+(bc+ca+ab)+abc}$$

$$= \frac{3+2(a+b+c)+(bc+ca+ab)}{1+(a+b+c)+(bc+ca+ab)+2+(a+b+c)}$$

$= 1$ (using (A)).

15. **(a) :** It is given that

$$D_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+2 & n^2+n \\ 2k-1 & n^2 & n^2+n+2 \end{vmatrix}$$

and $\sum_{k=1}^{n} D_k = 48$

We have to find out the value of n.

We know that $\sum_{k=1}^{n} 1 = n$

$\sum_{k=1}^{n} 2k = n^2 + n$

and $\sum_{k=1}^{n} (2k-1) = n^2$

$$\sum_{k=1}^{n} D_k = \begin{vmatrix} \sum_{k=1}^{n} 1 & n & n \\ \sum_{k=1}^{n} 2k & n^2+n+2 & n^2+n+2 \\ \sum_{k=1}^{n} (2k-1) & n^2 & n^2+n+2 \end{vmatrix}$$

$$\Rightarrow 48 = \begin{vmatrix} n & n & n \\ n^2+n & n^2+n+2 & n^2+n \\ n^2 & n^2 & n^2+n+2 \end{vmatrix}$$

$$= \begin{vmatrix} n & 0 & n \\ n^2+n & 2 & n^2+n \\ n^2 & 0 & n^2+n+2 \end{vmatrix} \quad [\text{By } C_2 \to C_2 - C_1]$$

$= 2(n^3 + n^2 + 2n - n^3) = 2(n^2 + 2n)$

$\Rightarrow 24 = n^2 + 2n$

$25 = (n+1)^2$

$n + 1 = 5$

$\therefore\ n = 4.$

16. **(b): Step-1:** Given:

$$\text{We have} \begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix}$$

$$= (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

We have to find out the value of k for which the given equation satisfy.

By applying the elementary row and column operations on the given determinant we will try to obtain the value required.

$$\text{Let } \Delta = \begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix} = x^k y^k z^k \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$$

Taking $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = x^k y^k z^k \begin{vmatrix} 1 & x^2 & x^3 \\ 0 & y^2-x^2 & y^3-x^3 \\ 0 & z^2-x^2 & z^3-x^3 \end{vmatrix}$$

Expand w.r.t. first column

$$\Delta = x^k y^k z^k \begin{vmatrix} y^2-x^2 & y^3-x^3 \\ z^2-x^2 & z^3-x^3 \end{vmatrix}$$

$$= x^k y^k z^k (x-y)(z-x) \begin{vmatrix} y+x & y^2+x^2+yx \\ z+x & z^2+x^2+zx \end{vmatrix}$$

$R_2 \to R_2 - R_1$

$$= x^k y^k z^k (y-x)(z-x) \begin{vmatrix} y+x & y^2+x^2+yx \\ z-y & z^2-y^2+x(z-y) \end{vmatrix}$$

$$= x^k y^k z^k (y-x)(z-x) \begin{vmatrix} y+x & y^2+x^2+yx \\ z-y & (z-y)(z+y+x) \end{vmatrix}$$

$$= x^k y^k z^k (y-x)(z-x)(z-y) \begin{vmatrix} y+x & y^2+x^2+yx \\ 1 & x+y+z \end{vmatrix}$$

$$= x^k y^k z^k (x-y)(y-z)(z-x)(xy+yz+zx)$$

$$= x^{k+1} y^{k+1} z^{k+1} (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

But given $\Delta = (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

Hence, $x^{k+1} y^{k+1} z^{k+1} = 1$

$\therefore \; k+1 = 0 \quad \therefore k = -1.$

17. (a) : It is given that $\begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix} = f(x) - xf'(x)$

We have to find out the value of $f(x)$.

We will try to make the maximum number of zero possible in any of the row or column to minimize the expansion.

We have, $\begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix} = f(x) - xf'(x)$

$$\Delta = \begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix} = \begin{vmatrix} \alpha & x-\alpha & x-\alpha & x-\alpha \\ x & \beta-x & 0 & 0 \\ x & 0 & \gamma-x & 0 \\ x & 0 & 0 & \delta-x \end{vmatrix}$$

By taking, $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$; $C_4 \to C_4 - C_1$

Expand w.r.t. first column

$$= \alpha \begin{vmatrix} \beta-x & 0 & 0 \\ 0 & \gamma-x & 0 \\ 0 & 0 & \delta-x \end{vmatrix} - x \begin{vmatrix} x-\alpha & x-\alpha & x-\alpha \\ 0 & \gamma-x & 0 \\ 0 & 0 & \delta-x \end{vmatrix}$$

$$+ x \begin{vmatrix} x-\alpha & x-\alpha & x-\alpha \\ \beta-x & 0 & 0 \\ 0 & 0 & \delta-x \end{vmatrix} - x \begin{vmatrix} x-\alpha & x-\alpha & x-\alpha \\ \beta-x & 0 & 0 \\ 0 & \gamma-x & 0 \end{vmatrix}$$

$= \alpha(\beta-x)(\gamma-x)(\delta-x) - x(x-\alpha)(\gamma-x)(\delta-x)$
$- x(\beta-x)(x-\alpha)(\delta-x) + x(\beta-x)(\gamma-x)(\alpha-x)$
$= (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) + x\,[(x-\beta)(x-\gamma)(x-\delta)$
$+ (x-\gamma)(x-\delta)(x-\alpha) + (x-\alpha)(x-\beta)(x-\delta)$
$+ (x-\alpha)(x-\beta)(x-\gamma)]$
$= f(x) - x f'(x)$

where, $f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$.

18. (c) : We have a quadratic eqn. $ax^2 + bx + c = 0$ with roots α, β and $S_n = \alpha^n + \beta^n$ for $n \geq 1$.

We have to find out the value of determinant

$$\Delta = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}.$$

We know that if α, β are the roots of quadratic equation $ax^2 + bx + c = 0$, then $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.

$\because \; S_n = \alpha^n + \beta^n$

$$\therefore \; \Delta = \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

$= \{\alpha\beta - (\alpha+\beta) + 1\}^2 \{(\alpha+\beta)^2 - 4\alpha\beta\}$

$$= \left(\frac{c}{a}+\frac{b}{a}+1\right)^2 \left(\frac{b^2}{a^2}-\frac{4c}{a}\right) = \frac{(a+b+c)^2(b^2-4ac)}{a^4}.$$

19. (d) : $f(x) = \begin{vmatrix} x & \cos x & e^{x^2} \\ \sin x & x^2 & \sec x \\ \tan x & 1 & 1 \end{vmatrix}$.

To find out the value of $\int_{-\pi/2}^{\pi/2} f(x)$.

If $f(-x) = f(x)$, then $f(x)$ is an even function,

$\therefore \; \int_{-\pi/2}^{\pi/2} f(x)dx = 2\int_0^{\pi/2} f(x)dx$

If $f(-x) = -f(x)$, then $f(x)$ is an odd function,

$\therefore \; \int_{-\pi/2}^{\pi/2} f(x)dx = 2\int_0^{\pi/2} f(x)dx$

$$\because \; f(x) = \begin{vmatrix} x & \cos x & e^{x^2} \\ \sin x & x^2 & \sec x \\ \tan x & 1 & 2 \end{vmatrix}$$

$$f(-x) = \begin{vmatrix} -x & \cos x & e^{x^2} \\ -\sin x & x^2 & \sec x \\ -\tan x & 1 & 2 \end{vmatrix} = -f(x)$$

$\therefore \; f(x)$ is an odd function

$\therefore \int_{-\pi/2}^{\pi/2} f(x)dx = 0$

20. **(d) :** The system of equations
$x - y\cos\theta + z\cos 2\theta = 0,$
$-x\cos\theta + y - z\cos\theta = 0,$
$x\cos 2\theta - y\cos\theta + z = 0$
has non-trivial solution.
To find the value of θ for which the given condition satisfy.
For the homogeneous system of equations, the non-trivial solution exist, only if Δ = 0.
Now to solve the determinant Δ, first we try to make the maximum number of zero possible in any of the row or column by applying the elementary row and column operations on the given determinant.
For non-trivial solution

$$\Delta = \begin{vmatrix} 1 & -\cos\theta & \cos 2\theta \\ -\cos\theta & 1 & -\cos\theta \\ \cos 2\theta & -\cos\theta & 1 \end{vmatrix} = 0$$

By $C_3 \to C_3 - C_1$

$$\Rightarrow \begin{vmatrix} 1 & -\cos\theta & \cos 2\theta - 1 \\ -\cos\theta & 1 & 0 \\ \cos 2\theta & -\cos\theta & 1-\cos 2\theta \end{vmatrix} = 0$$

By $R_3 \to R_3 + R_1$

$$\Rightarrow \begin{vmatrix} 1 & -\cos\theta & -2\sin^2\theta \\ -\cos\theta & 1 & 0 \\ 1+\cos 2\theta & -2\cos\theta & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & -\cos\theta & -2\sin^2\theta \\ -\cos\theta & 1 & 0 \\ 2\cos^2\theta & -2\cos\theta & 0 \end{vmatrix} = 0$$

Expand w.r.t. first row
$\Rightarrow -2\sin^2\theta\,(2\cos^2\theta - 2\cos^2\theta) = 0$
$\Rightarrow 0 = 0$ (true for all values of θ)

21. **(c) :** From definition of sum of two matrices if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two $m \times n$ matrices, then the sum $A + B$ of A and B is defined as $A + B = C$ where $C = [c_{ij}]_{m \times n}$ and $c_{ij} = a_{ij} + b_{ij}$ for each i and j.

$$A + B = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 3 & 2 \\ 3 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 4 \\ 1 & 3 & 2 \\ -1 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & -2+(-2) & 4+4 \\ 2+1 & 3+8 & 2+2 \\ 3+(-1) & 1+1 & 5+5 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 8 \\ 3 & 6 & 4 \\ 2 & 2 & 10 \end{bmatrix}.$$

22. **(a) :** A matrix is called a singular matrix if the value of its determinant is zero.

Given matrix is $A = \begin{bmatrix} \lambda & 7 & -2 \\ 4 & 1 & 3 \\ 2 & -1 & 2 \end{bmatrix}$

For A to be a singular matrix $|A| = 0$

$$\therefore \begin{vmatrix} \lambda & 7 & -2 \\ 4 & 1 & 3 \\ 2 & -1 & 2 \end{vmatrix} = 0$$

$\Rightarrow \lambda(2 + 3) - 7(8 - 6) - 2(-4 - 2) = 0$
$\Rightarrow 5\lambda - 14 + 16 = 0$
$\Rightarrow \lambda = -\frac{2}{5}.$

23. **(a) :** A square matrix A is called a skew symmetric matrix if $A^t = -A$
$(A + B)^t = A^t + B^t$ (property of transpose matrices)
$= -A - B = -(A + B)$
$\therefore (A + B)$ is a skew symmetric matrix.

24. **(c) :** Adjoint of a matrix is obtained by replacing each element of the matrix with its corresponding co-factor.

The given matrix is $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}$

$$\therefore \text{adj } A = \begin{bmatrix} 0 & 0 & 1 \\ -3 & -1 & 1 \\ -2 & -1 & -3 \end{bmatrix}$$

$$|\text{adj } A| = \begin{vmatrix} 0 & 0 & 1 \\ -3 & -1 & 1 \\ -2 & -1 & -3 \end{vmatrix}$$

$= 0 + 0 + 1\,(3 - 2) = 1.$

25. **(b):** $\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$. So, the rank = 3.

26. **(b)**

27. **(d) :** Since $|A| = 0$, therefore by definition of skew symmetric matrix, A is a skew symmetric matrix.

28. **(c):** Since $A + B$ is defined, $\therefore A, B$ are of the same order say $m \times n$.
Since AB is defined $\therefore n = m$
$\therefore A, B$ must be square matrices of the same type.

29. **(d):** Here $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$
$\Rightarrow x' = y, y' = x$
Hence reflection is on the line $y = x$.

30. **(a) :** This problem can be solved by simply comparing the L.H.S. elements by R.H.S. elements
i.e. $x + 1 = 3 \Rightarrow x = 2$
$y - 2 = 3 \Rightarrow y = 5.$

31. **(b) :** We have a matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

We have to determine the value of $A^2 - 5A + 6I$

We know that I is a unit matrix. We will use the definitions of multiplication, addition and subtraction of matrices to get the required result.

$$A^2 = A\,.\,A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times2+0\times2+1\times1 & 2\times0+0\times1-1\times1 & 2\times1+0\times3+1\times0 \\ 2\times2+1\times2+3\times1 & 2\times0+1\times1-3\times1 & 2\times1+1\times3+3\times0 \\ 1\times2-1\times2+0\times1 & 1\times10-1\times1+0\times(-1) & 1\times1-3\times1+0\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$\therefore\ A^2 - 5A + 6I$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5-5\times2+6\times1 & -1-5\times0+6\times0 & 2-5\times1+6\times0 \\ 9-5\times2+6\times0 & -2-5\times1+6\times1 & 5-5\times1+6\times0 \\ 0-5\times1+6\times0 & -1-5\times(-1)+6\times0 & -2-5\times0+6\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

32. **(d) :** $A = (a_{ij})$ is a scalar matrix of order $n \times n$ such that $a_{ij} = k$ for all i.

We have to find out the value of $|A|$.

$$A = \begin{bmatrix} a_{11} & 0 \ldots\ldots\ldots 0 \\ 0 & a_{22} \ldots\ldots 0 \\ \ldots\ldots\ldots\ldots & \ldots\ldots a_{nn} \end{bmatrix}$$

$$\therefore\ |A| = \begin{bmatrix} k & 0 \ldots\ldots 0 \\ 0 & k \ldots\ldots 0 \\ \ldots\ldots & \ldots\ldots \\ 0 & 0 \ldots\ldots k \end{bmatrix} = k^n.$$

33. **(a) :** We have a matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

We have to find out the value of A^n.

This problem can be solved by mathematical induction.

The given matrix is $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

First we calculate A^2,

$$A^2 = A\,.\,A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

and $A^3 = A^2\,.\,A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

and $A^4 = A^3\,.\,A = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}$

So by mathematical induction we can say, then,

$$A^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$$

34. **(b) :** The given matrix is $\begin{bmatrix} 2+x & 3 & 4 \\ 1 & -1 & 2 \\ x & 1 & -5 \end{bmatrix}$.

We have to calculate that value of x for which given matrix is a singular matrix.

A matrix is called a singular matrix if the value of its determinant is zero.

Given matrix is, $A = \begin{bmatrix} 2+x & 3 & 4 \\ 1 & -1 & 2 \\ x & 1 & -5 \end{bmatrix}$

From definition of singular matrix, $|A| = 0$

$$\therefore\ \begin{vmatrix} 2+x & 3 & 4 \\ 1 & -1 & 2 \\ x & 1 & -5 \end{vmatrix} = 0$$

$\Rightarrow\ (2 + x)(5 - 2) - 3(-5 - 2x) + 4(1 + x) = 0$

$\Rightarrow\ 6 + 3x + 15 + 6x + 4 + 4x = 0$

$\Rightarrow\ 13x = -25$

$\Rightarrow\ x = -25/13$

35. **(d) :** We have $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and $G(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$

We have to find out the value of $[F(\alpha)\,G(\beta)]^{-1}$

If A and B are two non-singular matrices then $(AB)^{-1} = B^{-1}A^{-1}$

We have apply the above property of matrices directly on our problem, but first it is necessary to know that the given matrices are non-singular or singular.

$$|F(\alpha)| = \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= \cos^2\alpha + \sin^2\alpha = 1 \neq 0$

$\therefore F(\alpha)$ is a non-singular matrix.

$$|G(\beta)| = \begin{vmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{vmatrix}$$

$= \cos^2\beta + \sin^2\beta = 1 \neq 0$

$\therefore G(\beta)$ is also a non-singular matrix.

So we can write directly,

$[F(\alpha)\, G(\beta)]^{-1} = [G(\beta)]^{-1}\, [F(\alpha)]^{-1}$

36. **(c) :** A matrix, $A(x) = (1-x)^{-1/2}\begin{pmatrix} 1 & -x \\ -x & 1 \end{pmatrix}$;

$-1 < x < 1$ and $z = \dfrac{x+y}{1+xy}$.

To find out the correct relation between $A(x)$, $A(z)$ and $A(y)$

If $F(x)$ = some expression involving x, then some other function such as $F(z)$ is obtained by replacing x by z in the expression.

We have, $A(x) = (1-x)^{-1/2}\begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}$

$$\therefore\ A(z) = A\left(\frac{x+y}{1+xy}\right)$$

$$= \left(1 - \frac{x+y}{1+xy}\right)^{-1/2}\begin{bmatrix} 1 & -\dfrac{x+y}{1+xy} \\ -\dfrac{x+y}{1+xy} & 1 \end{bmatrix}$$

$$= \left(\frac{1+xy-x-y}{1+xy}\right)^{-1/2}\begin{bmatrix} 1 & -\dfrac{x+y}{1+xy} \\ -\dfrac{x+y}{1+xy} & 1 \end{bmatrix}$$

$$= \left[\frac{1+xy}{(1-x)(1-y)}\right]^{1/2}\begin{bmatrix} 1 & -\dfrac{x+y}{1+xy} \\ -\dfrac{x+y}{1+xy} & 1 \end{bmatrix} \quad \text{...(i)}$$

Again $A(x)\,A(y)$

$$= (1-x)^{-1/2}\begin{pmatrix} 1 & -x \\ -x & 1 \end{pmatrix}.(1-y)^{-1/2}\begin{pmatrix} 1 & -y \\ -y & 1 \end{pmatrix}$$

$$= \left[\frac{1}{(1-x)(1-y)}\right]^{1/2}\begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}\begin{bmatrix} 1 & -y \\ -y & 1 \end{bmatrix}$$

$$= \left[\frac{1}{(1-x)(1-y)}\right]^{1/2}\begin{bmatrix} 1+xy & -(x+y) \\ -(x+y) & 1+xy \end{bmatrix}$$

$$= \left[\frac{(1+xy)}{(1-x)(1-y)}\right]^{1/2}\sqrt{1+xy}\begin{bmatrix} 1 & -\dfrac{x+y}{1+xy} \\ -\dfrac{x+y}{1+xy} & 1 \end{bmatrix}$$

$$\therefore\ A(x)\,A(y) \div \sqrt{1+xy}$$

$$= \left[\frac{1+xy}{(1-x)(1-y)}\right]^{1/2}\begin{bmatrix} 1 & -\dfrac{x+y}{1+xy} \\ -\dfrac{x+y}{1+xy} & 1 \end{bmatrix} = A(z)$$

(from (i))

Hence, $A(z) = A(x)\,A(y) \div \sqrt{1+xy}$

37. **(b) :** It is given that B is a non-singular matrix and A is a square matrix.

We have to find out the value of det $(B^{-1}AB)$.

For non-singular square matrixes

det (AB) = det (A) det (B) = det (B) det (A)

det $(B^{-1}AB)$ = det (B^{-1}) . det (A) . det (B)

= det (B^{-1}) . det (B) . det (A)

= det $(B^{-1}B)$. det (A) = 1 . det (A)

$\because$ det $(I) = 1$

= det A

38. **(d) :** $\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

From definition of product of two matrices $A = [a_{ij}]_{m \times n}$ be a matrix or order $m \times n$ and $B = [b_{jk}]_{n \times p}$ a matrix of order $n \times p$. Then the product AB is denoted as matrix $C = [c_{ik}]_{m \times p}$ of order $m \times p$

where, $c_{ik} = \sum\limits_{j=1}^{n} a_{ij}b_{jk}$.

$$\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\frac{\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}}{1+\tan^2\theta} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\text{or} \begin{bmatrix} 1-\tan^2\theta & -2\tan\theta \\ 2\tan\theta & 1-\tan^2\theta \end{bmatrix} \div \sec^2\theta = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

comparing on both sides,

$\Rightarrow\ a = \cos 2\theta,\ b = \sin 2\theta$

39. **(b) :** $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ & $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$

We have to find out the relation between a, b, α and β

$$A^2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix}$$

Therefore, $\begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$

comparing on both sides we get, $a^2 + b^2 = \alpha$ and $2ab = \beta$

40. (b) : $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$= (1 + abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$ (given)

$\Rightarrow 1 + abc = 0$ since the determinant $\neq 0$.

41. (c) : $\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$

$\Rightarrow - bc - 2a (c - b) + a (4c - 3b) = 0$

$\Rightarrow 2ac = b (a + c) \Rightarrow b = \dfrac{2ac}{a+c}$

$\therefore a, b, c$ are in H.P.

42. (a) : If n is a multiple of 3, then all the rows are identical and $\Delta = 0$

If n is not a multiple of 3, then $1 + \omega^n + \omega^{2n} = 0$ since $1 + \omega + \omega^2 = 0$

Now $C_1 \to C_1 + C_2 + C_3$ gives $\Delta = \begin{vmatrix} 0 & \omega^n & \omega^{2n} \\ 0 & \omega^{2n} & 1 \\ 0 & 0 & \omega^n \end{vmatrix}$

$\Rightarrow \Delta = 0$ for all integer n.

43. (b) : $\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

44. (b) : $AA^{-1} = I$

If R_1 of A is multiplied by C_3 of A^{-1}, we get $2 - \alpha + 3 = 0 \Rightarrow \alpha = 5$.

45. (c) : $A - A^2 = I \Rightarrow A (I - A) = I$

$\Rightarrow A^{-1} = I - A$.

46. (c) : $C_1 \to C_1 + (C_2 + C_3)$ gives

$f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix}$

$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix} = (x - 1)^2$.

47. (b) : $\Delta = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = \alpha^3 - 3\alpha + 2 = 0$

$\Rightarrow (\alpha - 1) (\alpha^2 + \alpha - 2) = 0$

$\Rightarrow (\alpha - 1)^2 (\alpha + 2) = 0$

$\Rightarrow \alpha = 1$ or $- 2$

$\alpha = 1 \Rightarrow x + y + z = 0$ giving infinite number of solutions.

$\alpha = - 2 \Rightarrow$ no solution.

48. (b) : $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 2b \\ 3a & 4b \end{bmatrix}$

$BA = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3b & 4b \end{bmatrix}$

$AB = BA \Rightarrow a = b = 1, 2, 3, \dots$.

49. (d) : The characteristic equation is

$\det\begin{bmatrix} -1-\lambda & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2}-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 + \dfrac{\lambda}{2} + \dfrac{1}{4} = 0$

$\therefore A^2 + \dfrac{A}{2} + \dfrac{I}{4} = 0 \Rightarrow A^2 = -\dfrac{A}{2} - \dfrac{I}{4}$

$\Rightarrow A^3 = -\dfrac{A^2}{2} - \dfrac{A}{4} = -\dfrac{1}{2}\left(-\dfrac{A}{2} - \dfrac{I}{4}\right) - \dfrac{A}{4} = \dfrac{I}{8}$.

50. (b) : $A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} - \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix}$

$= n\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - (n-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = nA - (n - 1) I$.

51. (d) : $A^2 - B^2 = (A - B) (A + B)$

$= A^2 + AB - BA - B^2$

$\therefore AB = BA$.

52. (a) : $a_n = a_1 r^{n-1}$, r is the common ratio

$\log a_n = \log a_1 + (n - 1) \log r$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ in Δ gives

$\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ 3\log r & 3\log r & 3\log r \\ 6\log r & 6\log r & 6\log r \end{vmatrix} = 0$

since R_2 and R_3 are proportional.

53. (d) : The determinant of the coefficient matrix is the product

$$\begin{vmatrix} 0 & 1 & 1 \\ 0 & a+b & c+d \\ 0 & ab & cd \end{vmatrix}\begin{vmatrix} 0 & 0 & 0 \\ 1 & c+d & cd \\ 1 & a+b & ab \end{vmatrix} = 0$$

for all a, b, c, d.

54. (b) : $-a + bx - cx = 0,\ -ay - b + cy = 0,\ az - bz - c = 0$

Eliminating a, b, c

$$\begin{vmatrix} -1 & x & -x \\ -y & -1 & y \\ z & -z & -1 \end{vmatrix} = 0$$

$\Rightarrow -(1 + yz) - x(y - yz) - x(yz + z) = 0$

$\Rightarrow xy + yz + zx = -1.$

55. (b) : $\begin{vmatrix} -1 & a & a \\ b & -1 & b \\ c & c & -1 \end{vmatrix} = 0$

$C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$ gives

$$\Rightarrow \begin{vmatrix} -1 & a+1 & a+1 \\ b & -(b+1) & 0 \\ c & 0 & -(1+c) \end{vmatrix} = 0$$

$R_1 \to \dfrac{R_1}{a+1},\ R_2 \to \dfrac{R_2}{b+1},\ R_3 \to \dfrac{R_3}{c+1}$ gives

$$\Rightarrow \begin{vmatrix} -\dfrac{1}{a+1} & 1 & 1 \\ \dfrac{b}{b+1} & -1 & 0 \\ \dfrac{c}{c+1} & 0 & -1 \end{vmatrix} = 0$$

Expanding by C_1, $-\dfrac{1}{a+1} + \dfrac{b}{b+1} + \dfrac{c}{c+1} = 0$

$$-\frac{1}{a+1} + 1 - \frac{1}{b+1} + 1 - \frac{1}{c+1} = 0$$

$$\therefore \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2.$$

56. (a) : $\begin{vmatrix} a & 4 & 1 \\ b & 3 & 1 \\ c & 2 & 1 \end{vmatrix} = 0 \Rightarrow a - 2b + c = 0$

a, b, c are in A.P.

57. (d) : $\Delta = \begin{vmatrix} n & n+1 & n+2 \\ n! & (n+1)! & (n+2)! \\ 1 & 1 & 1 \end{vmatrix}$

$C_3 \to C_3 - C_2,\ C_2 \to C_2 - C_1$ gives

$$= \begin{vmatrix} n & 1 & 1 \\ n! & n\cdot n! & (n+1)(n+!)! \\ 1 & 0 & 0 \end{vmatrix}$$

$= (n + 1)(n + 1)! - n \cdot n!$

$= [(n+ 1)^2 - n)\, n! = (n^2 + n + 1) \cdot n!.$

58. (c) : If A is singular there exists a non-zero vector X such that $AX = O$ *i.e.*,

$a_1x + a_2y + a_3z = 0$ ...(1)

$b_1x + b_2y + b_3z = 0$ ...(2)

$c_1x + c_2y + c_3z = 0$ ...(3)

Let $|x|$ be the max $\{|x|, |y|, |z|\}$

$(1) \Rightarrow -a_1 = a_2\dfrac{y}{x} + a_3\dfrac{z}{x}$

$$|-a_1| = \left|a_2\frac{y}{x} + a_3\frac{z}{x}\right| \le |a_2|\left|\frac{y}{x}\right| + |a_3|\left|\frac{z}{x}\right|$$

$\Rightarrow |a_1| \le |a_2| + |a_3|$, a contradiction.

Hence A is non-singular.

59. (d) : $\bar{A} = \begin{bmatrix} 23 & 1-i & -i \\ 1+i & -31 & 4+5i \\ -i & 4-5i & 17 \end{bmatrix}$

$(\bar{A})^T = A$

$\Rightarrow \det(\bar{A})^T = \det A$

$\Rightarrow \det \bar{A} = \det A$

$\Rightarrow \overline{\det A} = \det A$

$\Rightarrow \det A$ is real.

60. (b) : $\det A = (1 + a)(bc + b + c) - c - b$

$= abc\left(1 + \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right) = 0$

$\Rightarrow \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = -1.$

61. (d) : $R_3 \to R_1 + R_2 + R_3$ gives the determinant

$$(a + b + c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$ gives

$$= (a + b + c)^3\begin{vmatrix} 1 & 0 & 0 \\ 2b & -1 & 0 \\ 2c & 0 & 1 \end{vmatrix} = (a + b + c)^3.$$

$\because$ given matrix is singular.

$\therefore$ determinant $= 0$

$\Rightarrow (a + b + c)^3 = 0 \quad \Rightarrow \quad a + b + c = 0.$

62. (c) : $\Delta = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} + \begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$

For 1st determinant, $C_1 \leftrightarrow C_3$, then $C_3 \leftrightarrow C_2$

For 2nd determinant, $C_2 \leftrightarrow C_1$, then $C_2 \leftrightarrow C_3$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

$\therefore\ k = 2.$

63. **(d) :** $C_1 \to C_1 + C_2 + C_3$ gives

$$\Delta = (x+y)^2\begin{vmatrix}1 & x^2 & y^2\\1 & y^2 & 2xy\\1 & 2xy & x^2\end{vmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$\Delta = (x+y)^2\begin{vmatrix}1 & x^2 & y^2\\0 & y^2-x^2 & 2xy-y^2\\0 & 2xy-y^2 & x^2-2xy\end{vmatrix}$$

$$\Delta = (x+y)^2\begin{vmatrix}y^2-x^2 & 2xy-y^2\\2xy-x^2 & x^2-y^2\end{vmatrix}$$

$= -(x+y)^2\ [(x^2-y^2)^2 + (2xy-x^2)(2xy-y^2)]$

$= -(x+y)^2\ [x^2 - xy + y^2]^2$

$= -(x^3+y^3)^2.$

64. **(c) :** Expanding the two determinants,

$(1 - 3x^2 + 2x^3) + (3x^2 - x^3) = 0$

$\Rightarrow x^3 + 1 = 0 \Rightarrow x = -\omega, -\omega^2$

$x^{2007} + x^{-2007} = -1 - 1 = -2.$

65. **(a) :** $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$ gives

$$D = \begin{vmatrix}1 & 1 & 1\\0 & x & 0\\0 & 0 & y\end{vmatrix} = xy$$

which is divisible by both x and y.

66. **(a) :** Expanding by R_1, we get

$\Delta = \cos^2(\alpha+\beta) + \sin^2(\alpha+\beta) + \cos 2\beta$

$= 1 + \cos 2\beta.$

67. **(b) :** $\det(A^2) = (\det A)^2 = \begin{vmatrix}5 & 5\alpha & \alpha\\0 & \alpha & 5\alpha\\0 & 0 & 5\end{vmatrix}^2 = 25$

$$\Rightarrow (25\alpha)^2 = 25 \Rightarrow \alpha^2 = \frac{1}{25},\ |\alpha| = \frac{1}{5}.$$

68. **(a) :** Consider $a_n + a_{n+2} - 2a_{n+1} = A$(Say)

$$\therefore\ A = \int_0^{\pi/2}\left[\frac{(1-\cos 2nx) + 1 - \cos(2n+4)x - 2(1-\cos(2n+2)x)}{1-\cos 2x}\right]dx$$

$$A = \int_0^{\pi/2}\left[\frac{2\cos(2n+2)x - 2\cos(2n+2)x\cos 2x}{1-\cos 2x}\right]dx$$

$$A = \int_0^{\pi/2}\left[\frac{2\{\cos(2n+2)x\}(1-\cos 2x)}{1-\cos 2x}\right]dx$$

$$A = \int_0^{\pi/2} 2\cos(2n+2)x\,dx$$

$$A = 2\left(\frac{\sin(2n+2)x}{2n+2}\right)_0^{\pi/2} = 0$$

$\therefore\ a_n, a_{n+1}, a_{n+2} \in AP$, along with

$$a_1 = \int_0^{\pi/2}\frac{1-\cos 2x}{1-\cos 2x}dx \quad \therefore a_1 = \frac{\pi}{2}$$

$$\text{so } \Delta = \begin{vmatrix}\frac{\pi}{2} & a_2 & a_3\\a_4 & a_5 & a_6\\a_7 & a_8 & a_9\end{vmatrix} = \begin{vmatrix}a_1 & a_2 & a_3\\a_4 & a_5 & a_6\\a_7 & a_8 & a_9\end{vmatrix} \text{ Applying}$$

$C_1 \to C_1 + C_3 - 2C_2$

$$= \begin{vmatrix}0 & a_2 & a_3\\0 & a_5 & a_6\\0 & a_8 & a_9\end{vmatrix} = 0$$

69. **(c) :** $\Delta = \begin{vmatrix}a & 1 & 1\\1 & b & 1\\1 & 1 & c\end{vmatrix} = abc - (a+b+c) + 2$

As $\Delta > 0 \quad \therefore\ a.b.c + 2 > a + b + c$ ...(A)

Using $A.M. > G.M.$

$\Rightarrow \dfrac{a+b+c}{3} > (abc)^{1/3}$

or $a + b + c > 3(abc)^{1/3}$ ...(B)

By (A) and (B) we have

$\therefore\ abc + 2 > a + b + c > 3(abc)^{1/3}$

$\Rightarrow abc + 2 > 3(abc)^{1/3}$

$\Rightarrow 2 + x^3 > 3x \quad (x = (abc)^{1/3})$

$\Rightarrow x^3 - 3x + 2 > 0$

$\Rightarrow (x-1)^2(x+2) > 0 \quad \therefore x > -2$

$\because\ (abc)^{1/3} > -2$

$abc > -8$

70. **(d) :** Given $D_k = \begin{vmatrix}1 & 2k & 2k-1\\n & n^2+n+1 & n^2\\n & n^2+n & n^2+n+1\end{vmatrix}$ and

$$\sum_{k=1}^{n} D_k = 72$$

$$\therefore\ \sum_{k=1}^{n} D_k = \begin{vmatrix}\sum_{k=1}^{n} 1 & \sum_{k=1}^{n} 2k & \sum_{k=1}^{n} 2k-1\\n & n^2+n+1 & n^2\\n & n^2+n & n^2+n+1\end{vmatrix}$$

$$72 = \begin{vmatrix}n & n(n+1) & n^2\\n & n^2+n+1 & n^2\\n & n(n+1) & n^2+n+1\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$= \begin{vmatrix}n & n+1 & n^2\\0 & 1 & 0\\0 & 0 & n+1\end{vmatrix}$$

$\Rightarrow n(n+1) = 8(8+1) \Rightarrow n = 8$

71. **(c): Fact:** If A be the matrix of order m and B is matrix of cofactors of A then determinant of

B will be equal to $(m - 1)^{th}$ power of the determinant of A.

$|B| = |A|^{m-1} = (15)^{m-1}$

72. **(b):** As $A + B + C = \pi$

$\therefore$ $\tan(A + B + C) = \tan \pi = 0$

$\tan(A + B) = -\tan C$

$\cos(A + B) = -\cos C$

now the given determinant is skew symmetric matrix of order (odd) so its value is zero.

Fact. Determinant of skew symmetric matrix of odd order is always zero and determinant of skew symmetric matrix of even order is a non-zero perfect square number.

73. **(d):** Given $x + y + z = a$ ...(i)

$x + \omega y + \omega^2 z = b$ ...(ii)

$x + \omega^2 y + \omega z = c$ ...(iii)

by adding (i), (ii) and (iii) we get $x = \frac{a+b+c}{3}$

(Hence (a) is correct)

Again (i) + (ii) × ω^2 + (iii) × ω, we get (by adding)

$3y = a\omega^3 + b\omega^2 + c\omega$

$y = \frac{a + b\omega^2 + c\omega}{3} \Rightarrow$ (b) is correct

Similarly (i) + (ii) × ω + (iii) × ω^2 gives (by adding)

$z = \frac{a + b\omega + c\omega^2}{3} \Rightarrow$ (c) is correct.

74. **(a) :** Using $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$ we get

$$\begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0 \text{ (given)}$$

$\Rightarrow$ $(p - a)(q - b)r - (p - a)(c - r)b - (q - b)(a)(c - r) = 0$

$\Rightarrow$ $(p - a)(q - b)r + b(p - a)(r - c) + a(q - b)(r - c) = 0$

$$\Rightarrow \frac{r}{r-c} + \frac{b}{q-b} + \frac{a}{p-a} = 0$$

$$\Rightarrow \frac{r}{r-c} + \frac{q}{q-b} + \frac{p}{p-a} = 2$$

75. **(b) :**
$$\Delta = \begin{vmatrix} x & \frac{x(x-1)}{2} & \frac{x(x-1)(x-2)}{6} \\ y & \frac{y(y-1)}{2} & \frac{y(y-1)(y-2)}{6} \\ z & \frac{z(z-1)}{2} & \frac{z(z-1)(z-2)}{6} \end{vmatrix}$$

taking common x, y, z from R_1, R_2, R_3 and 2, 6 from C_2 and C_3.

$$= \frac{xyz}{12}\begin{vmatrix} 1 & x-1 & (x-1)(x-2) \\ 1 & y-1 & (y-1)(y-2) \\ 1 & z-1 & (z-1)(z-2) \end{vmatrix}$$

$R_1 \to R_1 - R_2$

$R_2 \to R_2 - R_3$

$$= \frac{xyz}{12}\begin{vmatrix} 0 & x-y & (x-1)(x-2)-(y-1)(y-2) \\ 0 & y-z & (y-1)(y-2)-(z-1)(z-2) \\ 1 & z-1 & (z-1)(z-2) \end{vmatrix}$$

$$= \frac{(xyz)(x-y)(y-z)(z-x)}{12}$$

76. **(d) :** Fact Adj (Adj A) = $|A|^{n-2}A$, If n is the order of the matrix and $|A| \neq 0$

$\therefore$ Adj (Adj A) = $|A|^{3-2}A$ $(\because n = 3)$

$\therefore$ $|\text{Adj (Adj } A)| = ||A|\,A|$

$= |A|^3 \det A$

$= |A|^4$ (Here $|A| = 11$)

$= 11^4 = 14641$

77. **(a) :**
$$\begin{vmatrix} \frac{\log x}{\log x} & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & \frac{4\log y}{\log y} & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & \frac{6\log z}{\log z} \end{vmatrix}$$

$$= \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & 4\log y & \log z \\ \log x & \log y & 6\log z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 6 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$

$$= \begin{vmatrix} 0 & 0 & -5 \\ 1 & 4 & 1 \\ 1 & 1 & 6 \end{vmatrix} = -5(1 - 4) = 15$$

78. **(d):** Each element of first column is the sum of two elements so

$\therefore$ Number of decomposition of $C_1 = 2 = l$ (say)

Number of decomposition of $C_2 = 3 = m$ (say)

Number of decomposition of $C_3 = 4 = n$ (say)

$\therefore$ Total number of decomposition of A are

$= l \times m \times n = 2 \times 3 \times 4 = 24$

79. **(b) :** $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4I_3$

$$\therefore A^n = 4^n(I_3)^n = 2^{2n}I_3 = \begin{pmatrix} 2^{2n} & 0 & 0 \\ 0 & 2^{2n} & 0 \\ 0 & 0 & 2^{2n} \end{pmatrix}$$

80. **(c):** Applying $R_1 \to R_1 + R_2 + R_3 + R_4$ we get each element in R_1 equals $(1 + a + b + c)$, which is common to all.

81. **(d):** Given $D = \begin{bmatrix} d_1 & 0 & 0 & 0 & ... & 0 \\ 0 & d_2 & 0 & 0 & ... & 0 \\ 0 & 0 & d_3 & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & ... & d_n \end{bmatrix}$

then $|D|$ = product of diagonal elements

$= d_1 d_2 \ldots . d_n$

Now cofactor of $D_{11} = d_2 d_3 \ldots . d_n$

Cofactor of $D_{22} = d_1 d_3 \ldots . d_n$

Cofactor $D_{ij}\ \forall\ i \neq j = 0$

and $D^{-1} = \frac{1}{|D|}$ (adj of D)

$$D^{-1} = \frac{1}{d_1 d_2 \ldots d_n}$$

$$= \begin{bmatrix} d_2 d_3 \ldots d_n & 0 & 0 & 0 \\ 0 & d_1 d_3 \ldots d_n & 0 & 0 \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ 0 & 0 & 0 & d_1 d_2 \ldots d_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{d_1} & 0 & 0 & 0 & ... & 0 \\ 0 & \frac{1}{d_2} & 0 & 0 & ... & 0 \\ 0 & 0 & \frac{1}{d_3} & 0 & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & \frac{1}{d_n} \end{bmatrix}$$

$= \text{diag}\,(d_1^{-1}, d_2^{-1}, d_3^{-1} \ldots .., d_n^{-1})$

Short Cut Method:

For a diagonal matrix $A = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & k_n \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ 0 & 0 & 0 & \frac{1}{k_n} \end{bmatrix}$$

$\therefore\ A^{-1} = [k_1^{-1}, k_2^{-1}, k_3^{-1} \ldots ., k_n^{-1}]$

$\therefore\ A^{-1} = \text{diag}\,(d_1^{-1}, d_2^{-1}, \ldots, d_n^{-1})$ replacing k_i by d_i, $i = 1, 2, \ldots n$

82. **(d):** It is well known fact due to definition of rank of a matrix.

$\therefore$ rank of $A = (a_{ij})_{m \times n}$ is $\leq$ minimum (m, n)

83. **(c):** Given matrix possesses a non zero minor of order 2 namely $\begin{pmatrix} 1 & -3 \\ 9 & 1 \end{pmatrix}$

$\therefore$ rank of $(A) \geq 2$ ...(i)

Now A does not possess any 3-rowed minor.

$\therefore\ r(A) \leq 2$ ....(ii)

By (i) and (ii) rank of $(A) = 2$

84. **(b):** Characteristic equation of A is given by

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & -4-\lambda & 2 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$$

$\Rightarrow\ (1 - \lambda)(-4 - \lambda)(7 - \lambda) = 0$

$\Rightarrow\ \lambda = 1, -4, 7$ are characteristics roots of A.

85. **(d):** Consider $A^2 = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$

But when $n = 2$ no choice amongs (a), (b), (c) are match with A^2.

86. **(b):** Given $A^T A = 1$

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sum a^2 & \sum ab & \sum ab \\ \sum cb & \sum b^2 & \sum bc \\ \sum ac & \sum ac & \sum c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \sum a^2 = 1$ and $\sum ab = \sum bc = \sum ca = 0$ ...(*)

Now $a^3 + b^3 + c^3 = (a + b + c)\,[a^2 + b^2 + c^2 - ab - bc - ca] + 3abc$

$\Rightarrow\ a^3 + b^3 + c^3 = (a + b + c)\,[1 - 0] + 3(1)$

$\Rightarrow\ a^3 + b^3 + c^3 = (a + b + c) + 3$

$\Rightarrow\ a^3 + b^3 + c^3 = 1 + 3$

$\because\ (a + b + c)^2 = \sum a^2 + 2\sum ab$

$a^3 + b^3 + c^3 = 4$

$(\sum a)^2 = 1$ using (*)

$\sum a = 1$

$a + b + c = 1$

Short Cut Method:

Given $A^T A = 1$

$\Rightarrow\ |A^T A| = 1\ \Rightarrow |A|^2 = 1$

now A is circulant matrix

$$\therefore\ \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = 1$$

$\Rightarrow\ a^3 + b^3 + c^3 - 3abc = 1$
$\Rightarrow\ a^3 + b^3 + c^3 = 1 + 3abc$
$= 1 + 3 = 4 \qquad (\because abc = 1)$

87. **(b) :** $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$

$= \cos x \begin{vmatrix} \sec x & 1 & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos x & \operatorname{cosec}^2 x \\ 1 & \cos x & \cos^2 x \end{vmatrix}$

Applying $C_2 \to C_2 - \cos x\, C_1$ and expanding we get

$f(x) = \cos x(\cos x - \cos^3 x)\left(\cos x - \frac{1}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right)$

$f(x) = -\cos^5 x - \sin^2 x$

$\therefore \int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} -\cos^5 x\, dx - \int_{-\pi}^{\pi} \sin^2 x\, dx$

$= -2\int_0^{\pi} \cos^5 x\, dx \ -2\int_0^{\pi} \sin^2 x\, dx$

$= 0 - 2 \times 2\int_0^{\pi/2} \sin^2 x\, dx = -4 \times \frac{1}{2} \times \frac{\pi}{2} = -\pi$

88. **(d) :** $A = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ where $t = \frac{2\pi}{3}$

$\therefore 3t = 2\pi$

$\therefore A^2 = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$

$A^2 = \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix}$

$A^2 \cdot A = A^3 = \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$

$= \begin{pmatrix} \cos 3t & -\sin 3t \\ \sin 3t & \cos 3t \end{pmatrix} = \begin{pmatrix} \cos 6\pi & -\sin 6\pi \\ \sin 6\pi & \cos 6\pi \end{pmatrix}$

$A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \therefore$ least value of $k = 3$.

89. **(c) :** Cofactor matrix of the matrix

$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = R(\text{say})$

Given cofactor matrix of R is $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$

Note: $a_1 \Delta = \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix}$ and $\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix} = c_1 \Delta$

But others are not equals to Δ.

90. **(b):** By given, we have

$-x^7 - x^6 + 0x^5 - 4x^4 + 8x^3 + 34x^2 + 75x + 21 =$
$px^7 + qx^6 + rx^5 + sx^4 + tx^3 + ux^2 + vx + w$

$\therefore$ on compairing both side we get

$p = q = -1$
$w = 21, v = 75$
$p = -1, t = 8$
$q = 0, t = -8$ which is not true.

Short Cut Method:

$\begin{vmatrix} x^2 + x & 2x - 1 & x + 3 \\ 3x + 1 & 2 + x^2 & x^3 - 3 \\ x - 3 & x^2 + 4 & 2x \end{vmatrix} = px^7 + qx^6 + rx^5 + sx^4 + tx^3 + ux^2 + vx + w \quad ...(*)$

For w, putting $x = 0$ both sides, we get

$\begin{vmatrix} 0 & -1 & 3 \\ 1 & 2 & -3 \\ -3 & 4 & 0 \end{vmatrix} = w \quad \therefore \quad w = 1(-9) + 3(10) = 21$

To get v, differentiate (*) w.r.t. x both sides and put $x = 0$

$\therefore \quad v = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & -3 \\ -3 & 4 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 3 \\ 3 & 0 & 0 \\ -3 & 4 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 3 \\ 1 & 2 & -3 \\ 1 & 0 & 2 \end{vmatrix}$

$v = (12 + 18 + 10) + (12 \times 3) + (5 - 6) = 75$

$\therefore w = 21, v = 75$

91. **(d) :** Given $\begin{vmatrix} 1 & m & \frac{m(m-1)}{2} \\ 1 & m+1 & \frac{(m+1)m}{2} \\ 1 & m+2 & \frac{(m+2)(m+1)}{2} \end{vmatrix}$

$(R_3 \to R_3 - R_2,\ R_2 \to R_2 - R_1)$

$= \begin{vmatrix} 1 & m & \frac{m(m-1)}{2} \\ 0 & 1 & m \\ 0 & 1 & m+1 \end{vmatrix} = m + 1 - m = 1$

92. **(a) :** From given

$f(x) = \sin x(4\sin^2 x - 1) - 1\,(2\sin x)$
$f(x) = 4\sin^3 x - 3\sin x$
$\therefore\ f(-x) = -4\sin^3 x + 3\sin x$
$f(x) = -f(x)$
$f(x)$ is an odd function

$\therefore \int_{-\pi/2}^{\pi/2} f(x)\, dx = 0$

93. **(d) :** $f'(x) = \begin{vmatrix} 1 & x^2 & 2x \\ \cos x & \tan x & \sin 2x \\ -\sin x & -x^3 & 5x \end{vmatrix}$

$+ \begin{vmatrix} x & 2x & 2x \\ \sin x & \sec^2 x & \sin 2x \\ \cos x & -3x^2 & 5x \end{vmatrix} + \begin{vmatrix} x & x^2 & 2 \\ \sin x & \tan x & 2\cos 2x \\ \cos x & -x^3 & 5 \end{vmatrix}$

Using $C_2 \to \frac{1}{x}C_2$ in 1st determinant,

$C_3 \to \frac{1}{x}C_3$ in 2nd determinant and

$C_2 \to \frac{1}{x}C_2$ in 3rd determinant

$$\therefore \frac{f'(x)}{x} = \begin{vmatrix} 1 & x & 2x \\ \cos x & \frac{\tan x}{x} & \sin 2x \\ -\sin x & -x^2 & 5x \end{vmatrix} + \begin{vmatrix} x & 2x & 2 \\ \sin x & \sec^2 x & \frac{\sin 2x}{x} \\ \cos x & -3x^2 & 5 \end{vmatrix} + \begin{vmatrix} x & x & \cos x \\ \sin x & \frac{\tan x}{x} & -x^2 \\ \cos x & 2\cos 2x & 5 \end{vmatrix}$$

$$\therefore \lim_{x \to 0} \frac{f'(x)}{x}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 5 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 5 \end{vmatrix}$$

$= 0 + 1(-2) + 1(-2) = -4$

94. **(a) :** $\therefore \ |A| = \begin{vmatrix} 0 & i-\sin x & i-\cos x \\ \sin x - i & 0 & \sin x - i \\ \cos x - i & i - \sin x & 0 \end{vmatrix}$

$= (i - \sin x)(\cos x - i)(\sin x - i)$
$\qquad - (i - \cos x)(i - \sin x)^2$

$= (i - \sin x)^2 (i - \cos x) - (i - \sin x)^2 (i - \cos x)$

$= 0$

Short Cut Method:

Quickly note that A is skew symmetric matrix of odd order and determinant of skew symmetric matrix of odd order is always equal to 0.

95. **(b):** As A is invertible *i.e.* $|A| \neq 0$

so A^{-1} exist

$\therefore$ rank (B) = rank $(A^{-1}(AB))$

rank $(B) \leq$ rank (AB)

$\because$ rank $(AB) <$ rank A ...(i)

Again rank $(AB) \leq$ rank B ...(ii)

By (i) and (ii)

rank $(AB) \geq$ rank $B \leq$ rank (AB)

$\Rightarrow$ Rank of (AB) = rank (B)

96. **(a) :** Since the system has a non-trivial solution

$$0 = \begin{vmatrix} p+a & b & c \\ a & q+b & c \\ a & b & r+c \end{vmatrix} = \begin{vmatrix} p & -q & 0 \\ 0 & q & -r \\ a & b & r+c \end{vmatrix}$$

using $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$.

Expanding along C_1, we get

$$p\begin{vmatrix} q & -r \\ b & r+c \end{vmatrix} + a\begin{vmatrix} -q & 0 \\ q & -r \end{vmatrix} = 0$$

$\Rightarrow p[q(r + c) + br] + aqr = 0$

$\Rightarrow pqr + pqc + prb + qra = 0$

$\Rightarrow \frac{a}{p} + \frac{b}{q} + \frac{c}{r} = -1.$

97. **(a) :** Solvig the first two equations, we obtain $x = 1$, $y = -1$. Putting this value in the last equation we obtain $\lambda - 2 = \mu$ or when $\lambda - \mu = 2$.

98. **(a) :** $|A| = (-1)^3 \begin{vmatrix} 0 & -\sin\alpha & -\sin\alpha\sin\beta \\ \sin\alpha & 0 & -\cos\alpha\cos\beta \\ \sin\alpha\sin\beta & \cos\alpha\cos\beta & 0 \end{vmatrix}$

[taking (-1) common from C_1, C_2 and C_3]

$= (-1)\ |A'| = -|A|$

$\Rightarrow 2|A| = 0 \Rightarrow |A| = 0 \Rightarrow A^{-1}$ does not exist

99. **(a) :** Let $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$,

then $A^2 = \begin{bmatrix} \alpha^2 + \beta\gamma & \beta\alpha + \beta\delta \\ \alpha\gamma + \gamma\delta & \beta\gamma + \delta^2 \end{bmatrix}$

$A^2 = O$

$\Rightarrow \alpha^2 + \beta\gamma = \gamma(\alpha + \delta) = \beta(\alpha + \delta) = \beta\gamma + \delta^2 = 0.$

$\Rightarrow \alpha^2 = \delta^2 \Rightarrow \alpha = \pm\delta$. If $\alpha = \delta \neq 0$

then $\gamma = 0$, $\beta = 0$. This gives us $\alpha^2 = \delta^2 = 0$

$\Rightarrow \alpha + \delta = 0.$

$\therefore$ If $\alpha = -\delta$, then $\alpha + \delta = 0$

which gives us that $\gamma(\alpha + \delta) = \beta(\alpha + \delta) = 0$ hold.

Thus, $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ where $\alpha^2 + \beta\gamma = 0$.

MCQs : More than One Correct Option

1. **(a, c) :** Given that, $AB = A$ and $BA = B$

$\therefore B^2 = BB \Rightarrow B^2 = (BA)B$ $[\because BA = B]$

$\Rightarrow B^2 = B(AB) \Rightarrow B^2 = BA$ $[\because AB = A]$

$\Rightarrow B^2 = B$ $[\because BA = B]$

2. **(a, d) :** Given that, $a_{ij} = i^2 - j^2$

$\therefore a_{ji} = j^2 - i^2 \Rightarrow a_{ij} = -a_{ji}$

Thus, A is a skew-symmetric matrix of even order. We know that the determinant of every skew symmetric matrix of even order is a perfect square and that of the odd order is zero. Hence, options (a) and (d) are correct.

3. **(a, c) :** Given that, $|AB| = |A||B| \neq 0$

$[\because |A| \neq 0, |B| \neq 0]$

Therefore, AB is a non-singular matrix. Consequently,

$(AB)^{-1}$ exists and $(AB)^{-1} = B^{-1}A^{-1}$.

4. **(a, c, d) :** We know that, $(A^T)^{-1} = (A^{-1})^T$

So, option (a) is true. In option (b), A^{-1} is a matrix and $|A|^{-1}$ is a number.

So, it is not true.
Now, $A^2(A^{-1})^2 = (AA)(A^{-1}A^{-1})$
$\Rightarrow A^2(A^{-1})^2 = A(AA^{-1})A^{-1} = (AI)A^{-1} = AA^{-1} = I$
$\therefore (A^2)^{-1} = (A^{-1})^2$
So, option (c) is true.
Since $AA^{-1} = I$
$\therefore |A||A^{-1}| = 1$
$\therefore |A^{-1}| = |A|^{-1}$ ($\because$ A is an invertible matrix)

5. **(a, b) :** Given that, $B = -A^{-1}BA$
$\Rightarrow AB = -A(A^{-1}BA) \Rightarrow AB = -((AA^{-1})(BA))$
$\Rightarrow AB = -(I(BA)) \Rightarrow AB = -BA$ .....(i)
$\Rightarrow AB + BA = O$
Now, $(A+B)^2 = (A+B)(A+B)$
$\Rightarrow (A+B)^2 = A^2 + AB + BA + B^2$
$\Rightarrow (A+B)^2 = A^2 + O + B^2$ [From eq.(i)]
$\Rightarrow (A+B)^2 = A^2 + B^2$.

6. **(c, d) :** Given that, $|B| = 64$
$\Rightarrow |\text{adj } A| = 64$
$\Rightarrow |A|^2 = 64$ ($\because |\text{adj } A| = |A|^{n-1}$)
$\Rightarrow |A| = \pm 8$.

7. **(a, b) :** (a) Put $y = x$, we have

$$\Delta = \begin{vmatrix} x & 2x-z & -z \\ x & 2x-z & -z \\ x & 2x-z & -z \end{vmatrix}.$$

All the rows are identical, so $\Delta = 0$.
$\Rightarrow (x-y)$ is a factor of Δ. Hence, option (a) is true.
(b) Put $x = y$ in 2nd row, $\Delta = 0$, so $(x-y)$ is a factor.
Put $x = y$ in 3rd row, $\Delta = 0$, so $(x-y)$ is a factor.
$\Rightarrow (x-y)^2$ is a factor of Δ, Hence, option (b) is true.
(c) $(x-y)^3$ can't be a factor because the highest power $(x-y)$ is $(x-y)^2$. Hence, option (c) is ruled out.

(d) $$\Delta = \begin{vmatrix} x & 2y-z & -z \\ y & 2x-z & -z \\ y & 2y-z & -z \end{vmatrix} = \begin{vmatrix} x & 2y & -z \\ y & 2x & -z \\ y & 2y & -z \end{vmatrix} + \begin{vmatrix} x & -z & -z \\ y & -z & -z \\ y & -z & -z \end{vmatrix}$$

$$= -2z\begin{vmatrix} x & y & 1 \\ y & x & 1 \\ y & y & 1 \end{vmatrix} + 0 = -2z(x-y)^2$$

$\therefore \Delta$ depends on z.
So option (d) is ruled out.

8. **(b, d) :** Applying $C_1 \to C_1 - \frac{\cos\phi}{\sin\phi}\cdot C_2$, we have

$$\Delta = \begin{vmatrix} 0 & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta/\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= \frac{-\sin\theta}{\sin\phi}[-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi]$$

$\Rightarrow \Delta = \sin\theta$ *i.e.*, independent of ϕ.

$$\therefore \left(\frac{d\Delta}{d\theta}\right)_{\theta=\pi/2} = (\cos\theta)_{\theta=\pi/2} = \cos\pi/2 = 0.$$

9. **(a, b) :** Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$\Delta = \begin{vmatrix} -d & -d & a+2d \\ -d & 2d & a \\ 2d & -d & a+d \end{vmatrix} = -9(a+d)d^2$$

which depends on a and d because $\frac{d\Delta}{da} \neq 0$ and $\frac{d\Delta}{dd} \neq 0$.

10. **(a, b, c) :** On expanding, we get

$$\Delta = \underset{\text{I}}{\underline{(a+b-x)}}\,\underset{\text{II}}{\underline{[x^2 + (a+b)x + (a^2+b^2-ab)]}}$$

Hence, options (a) and (b) are true.
If $a = b$, the discriminant of $x^2 + (a+b)x + (a^2+b^2-ab) = 0$ is $6a^2 - 6a^2 = 0$ and II part has only real root. Thus total number of real roots for $a = b$ is two.
Hence, option (c) is true.

11. **(a, b, d) :** Applying $C_1 \to C_1 + C_2 + C_3$ we obtain

$$\Delta = (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$

Applyig $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we obtain

$$\Delta = (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x-x^2 \\ 0 & x^2-x & 1-x^2 \end{vmatrix}$$

$= (1+x+x^2)\{(1-x)(1-x^2) + (x-x^2)^2\} = (1-x^3)^2$
$1-x^3$, $(1-x^3)^2$ are factors of Δ.
But $\Delta(x) = 0$ has just two real roots.
We have $\Delta'(x) = 2(1-x^3)(-3x^2) \Rightarrow \Delta'(1) = 0$.

12. **(a, b, d) :** We know that $\frac{d^n}{dx^n}\left[\frac{1}{x}\right] = \frac{(-1)^n n!}{x^{n+1}}$,
$\frac{d^n}{dx^n}[\log x] = \frac{(-1)^{n-1}(n-1)!}{x^n}$ and $\frac{d^n}{dx}[x^n] = n!$

$$\therefore f^n(x) = \begin{vmatrix} \frac{(-1)^n n!}{x^{n+1}} & \frac{(-1)^{n-1}(n-1)!}{x^n} & n! \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix}$$

Taking $(-1)^n n!$ common from R_1 and putting $x = 1$ we get,

$$f^n(1) = (-1)^n n! \begin{vmatrix} 1 & -1/n & (-1)^n \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix} = (-1)^n n!(0) = 0$$

Note that $f^n(1)$ is independent of a and n. Also, note that $y = a(x - f^n(1))$ *i.e.*, $y = ax$ is a straight line through the origin

13. **(a, d) :** Using $C_1 \to C_1 - b_3C_3$ and $C_2 \to C_2 - a_2C_3$ and then taking a_1 common from C_2 we obtain

$$\Delta = a_1 \begin{vmatrix} b_1x_1^2 + b_2x_1y_1 & x_1y_1 & y_1^2 \\ b_1x_2^2 + b_2x_2y_2 & x_2y_2 & y_2^2 \\ b_1x_3^2 + b_2x_3y_3 & x_3y_3 & y_3^2 \end{vmatrix}$$

using $C_1 \to C_1 - b_2C_2$ we get,

$$= a_1b_1 \begin{vmatrix} x_1^2 & x_1y_1 & y_1^2 \\ x_2^2 & x_2y_2 & y_2^2 \\ x_3^2 & x_3y_3 & y_3^2 \end{vmatrix}$$

$$= a_1\, b_1\, x_1^2\, x_2^2\, x_3^2 \begin{vmatrix} 1 & y_1/x_1 & (y_1/x_1)^2 \\ 1 & y_2/x_2 & (y_2/x_2)^2 \\ 1 & y_3/x_3 & (y_3/x_3)^2 \end{vmatrix}$$

$$= a_1\, b_1\, x_1^2\, x_2^2\, x_3^2 \left(\frac{y_2}{x_2} - \frac{y_1}{x_1}\right)\left(\frac{y_3}{x_3} - \frac{y_1}{x_1}\right)\left(\frac{y_3}{x_3} - \frac{y_2}{x_2}\right)$$

$$= a_1b_1\,(x_1y_2 - x_2y_1)(x_1y_3 - x_3y_1)(x_2y_3 - x_3y_2)$$

14. **(a, b, c) :** Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ we obtain

$$\Delta = \begin{vmatrix} a_1 - b_1 & a_1 - b_2 & a_1 - b_3 \\ a_2 - a_1 & a_2 - a_1 & a_2 - a_1 \\ a_3 - a_1 & a_3 - a_1 & a_3 - a_1 \end{vmatrix} = 0,$$

[$\because$ R_2 and R_3 are proportional].

15. **(a, b) :** We have

$$\frac{d\Delta}{dx} = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\alpha) & a + x\sin\alpha \\ \cos(x+\beta) & \cos(x+\beta) & b + x\sin\beta \\ \cos(x+\gamma) & \cos(x+\gamma) & c + x\sin\gamma \end{vmatrix} + \begin{vmatrix} \sin(x+\alpha) & -\sin(x+\alpha) & a + x\sin\alpha \\ \sin(x+\beta) & -\sin(x+\beta) & b + x\sin\beta \\ \sin(x+\gamma) & -\sin(x+\gamma) & c + x\sin\gamma \end{vmatrix} + \begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & \sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & \sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & \sin\gamma \end{vmatrix}$$

In the first two determinants, C_1 and C_2 are identical. Therefore, each of them is equal to zero. Thus,

$$\frac{d\Delta}{dx} = \begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & \sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & \sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & \sin\gamma \end{vmatrix}$$

Applying $C_1 \to C_1 - \cos x\, C_3$ and $C_2 \to C_2 + \sin x\, C_3$, we get

$$\frac{d\Delta}{dx} = \begin{vmatrix} \sin x\cos\alpha & \cos x\cos\alpha & \sin\alpha \\ \sin x\cos\beta & \cos x\cos\beta & \sin\beta \\ \sin x\cos\gamma & \cos x\cos\gamma & \sin\gamma \end{vmatrix} = 0,$$

because C_1 and C_2 are proportional. Therefore, Δ is independent of x and $\Delta = k$, where k is a constant.

Note that, $y = k$ is a straight line parallel to x-axis.

As $f(x) = \Delta = k$, is a constant, $f(x)$ cannot have a local maximum or minimum.

16. **(a, c) :** Expanding along C_3, the given determinant equals

$$(1)\begin{vmatrix} \sin\beta & \cos\beta \\ \sin\gamma & \cos\gamma \end{vmatrix} - (1)\begin{vmatrix} \sin\alpha & \cos\alpha \\ \sin\gamma & \cos\gamma \end{vmatrix} + (1)\begin{vmatrix} \sin\alpha & \cos\alpha \\ \sin\beta & \cos\beta \end{vmatrix}$$

$= (\sin\beta\cos\gamma - \sin\gamma\cos\beta) - (\sin\alpha\cos\gamma - \sin\gamma\cos\alpha)$
$+ (\sin\alpha\cos\beta - \sin\beta\cos\alpha)$

$= \sin(\beta - \gamma) + \sin(\gamma - \alpha) + \sin(\alpha - \beta)$,

proving the answer (c). but (a) is also correct because, by putting $A = \beta - \gamma$, $B = \gamma - \alpha$ and $C = \alpha - \beta$, we have $A + B + C = 0$,

so that $\sin(\beta - \gamma) + \sin(\gamma - \alpha) + \sin(\alpha - \beta)$

$= \sin A + \sin B + \sin C$

$$= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2\sin\left(-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$[A + B = -C]$

$$= -2\sin\frac{C}{2}\left(\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right)$$

$$= -2\sin\frac{C}{2}\left(\sin\frac{A}{2}\sin\frac{B}{2}\right)$$

$$= -4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= -4\sin\left(\frac{\beta-\gamma}{2}\right)\sin\left(\frac{\gamma-\alpha}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).$$

17. **(a, b, c) :** Multiplying C_3 by $\sin\beta$, the given determinant is

$$\Delta = \frac{1}{\sin\beta}\begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha\sin^2\beta \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha\sin^2\beta \\ 0 & -\sin\beta & \sin\beta\cos\beta \end{vmatrix}$$

Then $C_3 \to C_3 + (\cos\beta)C_2$ gives

$$\Delta = \frac{1}{\sin\beta}\begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha(\cos^2\beta + \sin^2\beta) \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha(\cos^2\beta + \sin^2\beta) \\ 0 & -\sin\beta & 0 \end{vmatrix}$$

$$= \frac{1}{\sin\beta}\begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha \\ 0 & -\sin\beta & 0 \end{vmatrix}$$

Finally, expanding along R_3, the determinant is

$$\Delta = \frac{1}{\sin\beta}(\sin\beta)\begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix}$$

$= \cos^2\alpha + \sin^2\alpha = 1.$

18. (a, b, c, d) : Expanding along C_1 we get

$$f(x) = 2\cos x\begin{vmatrix} 2\cos x & 1 \\ 1 & 2\cos x \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & 2\cos x \end{vmatrix}$$

$= 2\cos x\,(4\cos^2 x - 1) - 2\cos x$

$= 4\cos x\cos 2x = 2\,[\cos 3x + \cos x]$

Now, $\int_0^{\pi} f(x)dx = 2\int_0^{\pi}[\cos 3x + \cos x]\,dx$

$$= \left(\frac{2}{3}\sin 3x + 2\sin x\right)\Big]_0^{\pi} = 0$$

Maximum value of $f(x)$ is 4 corresponding to $x = 0$.

Next, $\lim_{x\to\pi/2} f(x) = 4\cos(\pi/2)\cos(\pi) = 0$

Also, $f'(x) = 2[-3\sin 3x - \sin x]$

$\Rightarrow f'(0) = 0.$

19. (a, b, c) : Multiply C_1 by a and apply $C_1 \to C_1 + bC_2 + cC_3$ to obtain

$$\Delta = \frac{1}{a}\begin{vmatrix} -a + b\cos C + c\cos B & \cos C & \cos B \\ a\cos C - b + c\cos A & -1 & \cos A \\ a\cos B + b\cos A - 2c & \cos A & -2 \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ -c & \cos A & -2 \end{vmatrix} \text{[using projection formula]}$$

$$= -\frac{c}{a}\,[\cos C\cos A + \cos B]$$

which is dependent of any of A, B, C.

20. (a, b, c) : Using in the change of base formula, $\log_\alpha\beta = \log\beta/\log\alpha$, we can write,

$$\Delta = \frac{1}{\log x\log y}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \sin(x+y) & -\cos(x+y) & \sin^2 z \end{vmatrix} = 0,$$

[$\because$ R_1 and R_2 are identical]

21. (b, c) : Expanding along R_1 we get

$$\Delta = x\begin{vmatrix} x & -\cos\theta\cos\phi \\ \cos\theta\cos\phi & x \end{vmatrix}$$

$$+\sin\theta\begin{vmatrix} \sin\theta & -\cos\theta\cos\phi \\ \cos\theta\sin\phi & x \end{vmatrix}$$

$$-\cos\theta\sin\phi\begin{vmatrix} \sin\theta & x \\ \cos\theta\sin\phi & \cos\theta\cos\phi \end{vmatrix}$$

$= x[x^2 + \cos^2\theta\cos^2\phi] + \sin\theta[x\sin\theta + \cos^2\theta\cos\phi\sin\phi]$

$-\cos^2\theta\sin\phi\,[\sin\theta\cos\phi - x\sin\phi]$

$= x^3 + x\,[\cos^2\theta\cos^2\phi + \sin^2\theta + \sin^2\phi\cos^2\theta]$

$= x^3 + x$

which is independent of θ and ϕ.

22. (a, b, c) : Let $D_1 = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & z_1 \end{bmatrix}$,

and $D_2 = \begin{bmatrix} x_2 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & z_2 \end{bmatrix}$,

then $D_1D_2 = \begin{bmatrix} x_1x_2 & 0 & 0 \\ 0 & y_1y_2 & 0 \\ 0 & 0 & z_1z_2 \end{bmatrix}$ which is a diagonal matrix.

Similarly $D_1D_2 = D_2D_1$.

Also, both D_1^2, D_2^2 and hence $D_1^2 + D_2^2$ are diagonal matrices.

23. (a, b, c) : We have $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = B$ (say).

Now, $A^{-n} = B^n = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$

$\Rightarrow \frac{1}{n}A^{-n} = \begin{bmatrix} 1/n & 0 \\ -1 & 1/n \end{bmatrix}$

and $\frac{1}{n^2}A^{-n} = \begin{bmatrix} 1/n^2 & 0 \\ -1/n & 1/n^2 \end{bmatrix}$

$\Rightarrow \lim_{n\to\infty}\frac{1}{n}A^{-n} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

and $\lim_{n\to\infty}\frac{1}{n^2}A^{-n} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

24. (a, b, c, d) : We have $B * A = 1/2\,(BA + AB)$

$= 1/2\,(AB + BA) = A * B,$

$A * A = 1/2\,(AA + AA) = 1/2\,(A^2 + A^2) = A^2,$

$A * (B + C) = 1/2\,[A(B + C) + (B + C)A]$

$= 1/2\,[AB + AC + BA + CA]$

$= 1/2\,(AB + BA) + 1/2\,(AC + CA)$

$= A * B + A * C$

and $A * I = 1/2\,(AI + IA)$

$= 1/2(A + A) = 1/2(2A) = A.$

25. (a, b, c, d) : We have $A^2B = A(AB) = A(BA)$

$= (AB)A = (BA)A = BA^2,$

$A^3B = A(A^2B) = A(BA^2)$

$= (AB)A^2 = (BA)A^2 = BA^3$ etc.

In general, by induction $A^nB = BA^n$, $\forall\ n \in N$.

Since A and B commute, (b) and (c) hold.

Also, $(A^n - B^n)(A^n + B^n)$

$= A^nA^n - B^nA^n + A^nB^n - B^nB^n = A^{2n} - B^{2n}$,

[$\because$ $A^nB^n = B^nA^n$ as A and B commute].

26. **(a, b, c, d) :** $A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

We have $A^2 - 4A - 5I_3$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = O$$

$\Rightarrow$ $5I_3 = A^2 - 4A = A(A - 4I_3)$

$\Rightarrow$ $I_3 = A\left[\frac{1}{5}(A - 4I_3)\right]$ $\Rightarrow$ $A^{-1} = \frac{1}{5}(A - 4I_3)$

Note that $|A| = 5$. Since $|A^3| = |A|^3 = 5^3 \neq 0$, A^3 is invertible.

Similarly, A^2 is invertible.

27. **(a, b, c) :** Let $a_i = a_1 + (i - 1)d$.

We have $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_5 & a_6 & a_7 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ 3d & 3d & 3d \\ d & d & d \end{vmatrix} = 0$

[using $R_3 \to R_3 - R_2$, and $R_2 \to R_2 - R_1$]

$\Rightarrow$ A is singular.

$\therefore$ The given system of homogeneous equations has infinite number of solutions.

Also, $|B| = a_1^2 + a_2^2 \neq 0$.

Thus, B is non-singular.

28. **(b, d) :** $x + y = 1$ ...(1)

$(c + 2)\,x + (c + 4)\,y = 6$ ...(2)

$(c + 2)^2\,x + (c + 4)^2\,y = 36$ ...(3)

$(2) - (c + 2)\,(1) \to (2)$, $(3) - (c + 2)^2\,(1) \to (3)$ gives

$x + y = 1$ ...(1)

$2y = 4 - c$ ...(2)

$4\,(c + 3)\,y = (c + 8)\,(4 - c)$ ...(3)

$(3) - 2\,(c + 3)\,(2) \to (3)$ gives

$0 = [(c + 8) - 2\,(c + 3)]\,(4 - c)$

$\Rightarrow (4 - c)\,(2 - c) = 0 \Rightarrow c = 2,\ 4.$

29. **(a, c) :** $R_1 \to R_1 - R_2$, $R_2 = R_2 - R_3$ we get

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \sin^2\theta & \cos^2\theta & 1 + 4\sin 4\theta \end{vmatrix} = 0$$

Applying $C_2 \to C_2 + C_1$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ \sin^2\theta & 1 & 1 + 4\sin 4\theta \end{vmatrix} = 0$$

$\Rightarrow$ $\sin 4\theta = -1/2$

$4\theta = 210°, 330°$

$\theta = \frac{210°}{4}, \frac{330°}{4} = \frac{7\pi}{24}, \frac{11\pi}{24}$

30. **(b, c):** For non trivial solution, determinant of coefficient matrix of system of equation vanish

$$\therefore \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

solving the determinant we get

$\theta = m\pi$ or $\theta = n\pi + (-1)^n \frac{\pi}{6}$

Assertion and Reason

1. **(b) :** Statement-1 is $\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}\begin{pmatrix} d'_1 & 0 & 0 \\ 0 & d'_2 & 0 \\ 0 & 0 & d'_3 \end{pmatrix}$

$$= \begin{pmatrix} d_1d'_1 & 0 & 0 \\ 0 & d_2d'_2 & 0 \\ 0 & 0 & d_3d'_3 \end{pmatrix}$$

which is also a diagonal matrix. Thus Statement-2 is true but is not a correct explanation of Statement-1.

2. **(d) :** Let $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$

Now the other bracket cannot vanish in the light of the correct Statement-2.

$\therefore$ The system can have non-zero solutions (x, y, z are not all zero).

If $D = 0$ $\Rightarrow$ $a + b + c = 0$

$\Rightarrow$ Statement-1 is false.

3. **(a) :** Let $D(x) = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x)$

where $\lambda_1 = (b_2c_3 - b_3c_2)$, $\lambda_2 = a_2c_3 - a_3c_2$,

$\lambda_3 = a_2b_3 - a_3b_2$, then

$$\int D(x)\,dx = \int \lambda_1 f_1(x)\,dx + \int \lambda_2 f_2(x)\,dx + \int \lambda_3 f_3(x)\,dx + c \quad .....(i)$$

$$\int D(x)\,dx = \begin{vmatrix} \int f_1(x)\,dx & \int f_2(x)\,dx & \int f_3(x)\,dx \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + c$$

Thus Statement-1 is true and follows from Statement-2 which we have applied in equation (i)

4. **(d) :** The Statement-2 is true since
$AI = IA, AA^{-1} = A^{-1}A = I, A\,(\text{adj}\,A) = (\text{adj}\,A)\,A$
But a matrix can commute with general order matrices which may be infinite in number.
Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix which commute with A, then $AB = BA$.
$\Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$
$\Rightarrow \begin{bmatrix} a+2c & b+2d \\ -a-c & -b-d \end{bmatrix} = \begin{bmatrix} a-b & 2a-b \\ c-d & 2c-d \end{bmatrix}$
$\Rightarrow\ a + 2c = a - b, b + 2d = 2a - b, -a - c = c - d,$
$-b - d = 2c - d$
The above four relations are equivalent to only two independent relations :
$a - d = b, b + 2c = 0$
If $d = \lambda$, then $a = b + \lambda = -2c + \lambda$
Thus, $\begin{bmatrix} \lambda-2c & -2c \\ c & \lambda \end{bmatrix}$ are all possible 2×2 matrices which commute with the given matrix
$A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$
λ and c being any arbitary complex numbers, thus, Statement-1 is false.

5. **(c) :** The Statement-2 is false since
$\det A' = \det(-A')$ is not true.
Indeed, $\det(-A') = (-1)^3 \det A'$
Now as $A = -A'$ (A is skew-symmetric)
$\det A = \det(-A') = -\det(A') = -\det A$
$\Rightarrow\ \det A = 0$
Thus Statement-1 is true

6. **(a) :** If $A = \begin{bmatrix} 1 & -a & b \\ a & 1 & -c \\ -b & c & 1 \end{bmatrix}, (I-A) = \begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$
The system is $(I - A)X = O \Rightarrow X = O$ since $I - A$ is non-singular.

7. **(a) :** $(BX)^T(BY)$
$= ((I - A)(I + A)^{-1}X)^T(I - A)(I + A)^{-1}Y$
$= X^T(I - A)^{-1}(I + A)(I - A)(I + A)^{-1}Y$
$= X^T(I + A)(I - A)^{-1}(I - A)(I + A)^{-1}Y$
$= X^T \cdot I \cdot I \cdot Y = X^TY$
Since $A^T = -A$ and $(I - A)(I + A) = (I + A)(I - A)$

8. **(d) :** $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$
$\Rightarrow \lambda^3 + \lambda^2 + \lambda + 1 = 0 \Rightarrow A^3 + A^2 + A + I = 0$
$\Rightarrow A^3 + A^2 + A = -I.$

9. **(d) :**
$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow 5-11\lambda+7\lambda^2-\lambda^3 = 0$
$\therefore\ 5I - 11A + 7A^2 - A^3 = 0$
Multiplying by A^{-1}
$5A^{-1} = A^2 - 7A + 11I.$

10. **(b) :**
$\begin{vmatrix} x & -2 & 1 \\ 2 & x & -3 \\ -1 & 3 & x \end{vmatrix} = 0 \Rightarrow x(x^2+9)+2(2x-3)+6+x = 0$
$x^3 + 14x = 0 \Rightarrow x = 0, \pm\sqrt{14}\,i.$

11. **(a) :** $\because \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 9 & 8 & 7 \end{bmatrix}$ is singular therefore inverse does not exist.
$\left[\because \begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 9 & 8 & 7 \end{vmatrix} = 0, \text{ since } R_2 = 2R_1\right]$
Statement-1, statement-2 are both true and Statement-2 is the correct explanation of Statement-1.

Passage Comprehension

PASSAGE 1 :

Since, A is orthogonal $\Rightarrow AA' = I$

$$AA' = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+b^2+c^2 & ac+ab+bc & ab+bc+ca \\ ca+ab+bc & a^2+b^2+c^2 & cb+ba+ac \\ ab+bc+ca & bc+ca+ab & a^2+b^2+c^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(given } AA' = I\text{)}$$

$a^2 + b^2 + c^2 = 1$
Now, $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow\ (a+b+c)^2 = 1$

1. **(d) :** *i.e.*, $a+b+c = \pm 1$
2. **(a) :** $ab + bc + ca = 0$
3. **(d) :** Also, $abc = p$

   Now, $a^3 + b^3 + c^3 - 3abc$

   $= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

   $\Rightarrow\ a^3+b^3+c^3-3p = \pm 1\,(1-0)$

   $\therefore\ a^3+b^3+c^3 = 3p \pm 1\ \ (\because\ abc = p)$
4. **(d) :** Now, if a, b, c are the roots of cubic.

   $\therefore\ x^3-(a+b+c)x^2+(ab+bc+ca)x-abc = 0$

   $\Rightarrow\ x^3 \pm x^2 - p = 0$

PASSAGE 2 :

1. **(c) :** $A.\ \text{adj}\,A = |A|I$ ....(i)

$$= |A|\begin{bmatrix} 1 & 0 & 0 & \ldots\ldots \\ 0 & 1 & 0 & \ldots\ldots \\ 0 & 0 & 1 & \ldots\ldots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots 1 \end{bmatrix}_{n\times n}$$

$$= \begin{bmatrix} |A| & 0 & 0 & \ldots\ldots \\ 0 & |A| & 0 & \ldots\ldots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & |A| \end{bmatrix}_{n\times n}$$

Taking the determinants,

$$|A\cdot \text{adj}\,A| = \begin{vmatrix} |A| & 0 & 0 & 0 & \ldots \\ 0 & |A| & 0 & 0 & \ldots \\ 0 & 0 & |A| & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & |A| \end{vmatrix}_{n\times n}$$

$\Rightarrow\ |A|\cdot|\text{adj}\,A| = |A|^n$

$\Rightarrow\ |\text{adj}\,A| = \dfrac{|A|^n}{|A|} = |A|^{n-1}.$

2. **(b) :** $(\text{adj}\,A)(\text{adj}\,A)^{-1} = I\quad (\because\ AA^{-1} = I)$

   $\Rightarrow\ A(\text{adj}\,A)\,(\text{adj}\,A)^{-1} = AI$

   (pre multiplication by A)

   $\Rightarrow\ |A|I\,(\text{adj}\,A)^{-1} = A$

   $I(\text{adj}\,A)^{-1} = \dfrac{A}{|A|}\quad\Rightarrow\quad(\text{adj}\ A)^{-1} = \dfrac{A}{|A|}$
3. **(c) :** Replacing A by adj A in (i)

   $\Rightarrow$ adj $A\cdot$ adj adj A

   $= |\text{adj}\,A|\,I = |A|^{n-1}I$ ...(ii)

   Taking determinants of (ii)

   $|\text{adj adj}\,A| = \dfrac{(|A|^{n-1})^n}{|\text{adj}\,A|} = \dfrac{|A|^{n^2-n}}{|A|^{n-1}}$

   $= |A|^{n^2-2n+1} = |A|^{(n-1)^2}$
4. **(d) :** From (ii) $\Rightarrow$ adj adj $A = |A|^{n-1}(\text{adj}\,A)^{-1}$

   $= |A|^{n-2}A.$

PASSAGE 3 :

1. **(c) :** $\because\ AX = \lambda X\ \Rightarrow AXX^{-1} = \lambda XX^{-1}$

   (post multiplictaion by X^{-1})

   $\Rightarrow\ AI = \lambda I\ \ \Rightarrow\ A - \lambda I = O\ \ \Rightarrow\ |A-\lambda I| = 0$

   $\Rightarrow \begin{vmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{vmatrix} = 0$

   $\Rightarrow \lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

   $\Rightarrow (\lambda-3)(\lambda-6)(\lambda-9) = 0$

   $\lambda = 3, 6, 9.$
2. **(c) :** H.C.F. of 3, 6 and 9 = 3.
3. **(d) :** $3\cdot 6\cdot 9 = 162.$
4. **(d) :** $\lambda = 3 \Rightarrow 2x - 2y = 0$

   $-2x + 3y + 2z = 0$

   $\Rightarrow \dfrac{x}{-4} = \dfrac{y}{-4} = \dfrac{z}{2} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

   $\lambda = 6 \Rightarrow -x - 2y = 0$

   $-2x + 2z = 0$

   $\Rightarrow \dfrac{x}{-4} = \dfrac{y}{2} = \dfrac{z}{-4} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

   $\lambda = 9\ \ \Rightarrow -4x - 2y = 0$

   $-2x - 3y + 2z = 0$

   $\Rightarrow \dfrac{x}{-4} = \dfrac{y}{8} = \dfrac{z}{8} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

   The above 3 vectors are mutually orthogonal.

PASSAGE 4 :

1. **(d):** Characteristic equation of matrix $A = |A - \lambda I| = 0$

   $\Rightarrow \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$

   $\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

   $\therefore A^3 - 6A^2 + 11A - 6I = O$ ...(1)

   $\Rightarrow A^3 - 6A^2 + 10A - 5I = I - A.$
2. **(d) :** Multiplying (1) by A^{-1},

   $A^2 - 6A + 11I - 6A^{-1} = O$

   $\Rightarrow 6A^{-1} = A^2 - 6A + 11I$ ...(2)
3. **(b) :** Multiplying (1) by A,

   $A^4 - 6A^3 + 11A^2 - 6A = O$

   $\therefore A^4 = 6A^3 - 11A^2 + 6A$

   $= 6(6A^2 - 11A + 6I) - 11A^2 + 6A$

   $= 25A^2 - 60A + 36I$

4. **(a) :** Multiplying (2) by A^{-1},
$6A^{-2} = A - 6I + 11A^{-1}$
$36A^{-2} = 6A - 36I + 11(A^2 - 6A + 11I)$
$= 11A^2 - 60A + 85I.$

$\Rightarrow \; A^{-2} = \frac{1}{36}[11A^2 - 60A + 85I]$

PASSAGE 5 :

1. **(b) :** Characteristic equation of matrix A is $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1 - \lambda)[(1 - \lambda)^2 - 4] = 0$
$\Rightarrow (1 - \lambda)(\lambda^2 - 2\lambda - 3) = 0$
$\Rightarrow (1 - \lambda)(\lambda - 3)(\lambda + 1) = 0$
$\Rightarrow \lambda = 1, -1, 3.$

2. **(a) :** $A^2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+4 & 2+2 & 0 \\ 2+2 & 4+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Trace of $A = 1 + 1 + 1 = 3 = 3^1 + 1 + (-1)^1$
Trace of $A = 5 + 5 + 1 = 11 = 3^2 + 1 + (-1)^2$
[For $n = 2$, (a), (d) holds]

$A^3 = \begin{bmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 14 & 0 \\ 14 & 13 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Trace $A^3 = 27 = 3^3 + 1 + (-1)^3$
(only (a) holds for $n = 3$)

3. **(d) :** Since matrix A satisfies its characteristic equation
$\therefore \; (A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I) = 0$
$\Rightarrow A = \lambda_1 I, \lambda_2 I, \lambda_3 I.$

PASSAGE 6 :

1. **(b) :** $\text{Adj } A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}' = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$

$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}}{3}$

$(A^{-1})^2 = \frac{1}{9}\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 1 & -8 \\ 0 & 9 \end{pmatrix}$

$(A^{-1})^3 = \frac{1}{27}\begin{pmatrix} 1 & -8 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = \frac{1}{27}\begin{pmatrix} 1 & -26 \\ 0 & 27 \end{pmatrix}$

2. **(b) :** Since $A^2 = I \;\therefore\; A^{-1}A^2 = IA^{-1}$
$\Rightarrow A = A^{-1}$

3. **(b) :**

$$\text{Adj } D = \begin{pmatrix} d_2 d_3 \ldots d_n & 0 \ldots\ldots & 0 \\ 0 & d_1 d_3 \ldots\ldots d_n & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 \ldots\ldots & d_1 d_2 \ldots\ldots d_{n-1} \end{pmatrix}$$

$|D| = d_1 d_2 \ldots\ldots d_n$

$$\therefore \; D^{-1} = \frac{\text{Adj } D}{|D|} = \begin{pmatrix} d_1^{-1} & 0 \ldots\ldots & 0 \\ 0 & d_2^{-1} & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 \ldots\ldots & d_n^{-1} \end{pmatrix}$$

$= \text{diagonal}(d_1^{-1}, d_2^{-1}, \ldots\ldots, d_n^{-1})$

PASSAGE 7 :

1. **(b) :** Since A is invertible
$\therefore \; AA^{-1} = A^{-1}A = I$
$\therefore \; |AA^{-1}| = |A^{-1}A| = |I| = 1$
$\Rightarrow |A||A^{-1}| = |A^{-1}||A| = 1$
$\Rightarrow |A^{-1}| = \frac{1}{|A|}$ *i.e.*, $\det(A^{-1}) = \frac{1}{\det A}$

2. **(b) :** Since $AB = AC$
$\therefore \; A^{-1}(AB) = A^{-1}(AC)$ if A^{-1} exists
$\Rightarrow (A^{-1}A)B = (A^{-1}A)C$
$\Rightarrow IB = IC \qquad \Rightarrow B = C$
Thus A is non-singular if $AB = AC$
$\Rightarrow B = C$

3. **(d) :** Since $A^2 - A + I = 0$
$\therefore \; A^{-1}A^2 - A^{-1}A + A^{-1}I = A^{-1} \cdot 0 = 0$
$\Rightarrow A - I + A^{-1} = 0$
$\Rightarrow A^{-1} = I - A.$

PASSAGE 8 :

1. **(a) :** Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & K \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & K \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & -3 \\ 3 & -1 & K-3 \end{vmatrix}$

$C_3 \to C_3 - C_1, C_2 \to C_2 - C_1$
$= -K + 3 - 3 = -K$
Since the given system has a unique solution.
$\therefore \; |A| \neq 0 \;\therefore\; -K \neq 0$ *i.e.*, $K \neq 0$

2. **(a) :** $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

$$\text{Adj. } A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}' = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

$$(\text{Adj. } A)B = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4+10-6 \\ 8-20+12 \\ -4+10-6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$$

$\therefore$ the given system of equations has many solutions.

3. **(a) :** $|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & p+2 \\ 0 & 2p+1 & 1 \end{vmatrix}$

$= -(p + 2)(2p + 1)$

Since the given system is inconsistent.

$\therefore \quad |A| = 0$

$\Rightarrow \quad -(p + 2)(2p + 1) = 0$

$\Rightarrow p = -2, -(1/2) \qquad \therefore \ p = -2$

PASSAGE 9 :

1. **(c) :** We have

$x - cy - bz = 0$

$cx - y + az = 0$

$bx + ay - z = 0$

Since x, y, z are not all zero.

$\therefore$ The system will have a non-trivial solution.

$$\therefore \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$\Rightarrow 1 - a^2 + c(-c - ab) - b(ac + b) = 0$

$\Rightarrow 1 - a^2 - c^2 - abc - abc - b^2 = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$

2. **(c) :** Since the system has a non-trivial solution.

$$\therefore \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$\Rightarrow \lambda(-\cos^2\alpha - \sin^2\alpha) - (-\sin\alpha\cos\alpha - \sin\alpha\cos\alpha)$
$-1(\sin^2\alpha - \cos^2\alpha) = 0$

$\Rightarrow -\lambda + 2\sin\alpha\cos\alpha + \cos 2\alpha = 0$

$\Rightarrow \lambda = \cos 2\alpha + \sin 2\alpha$

$$= \sqrt{2}\cos\left(2\alpha - \frac{\pi}{4}\right)$$

Since $-1 \le \cos\left(2\alpha - \frac{\pi}{4}\right) \le 1 \ \forall \ \alpha \in R$

$\therefore \ -\sqrt{2} \le \lambda \le \sqrt{2} \quad$ *i.e.*, $\ \lambda \in [-\sqrt{2}, \sqrt{2}]$

3. **(c) :** For a non-trivial solution, we have

$$\begin{vmatrix} 1 & 4a & a \\ 1 & 3b & b \\ 1 & 2c & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 4a & a \\ 0 & 3b-4a & b-a \\ 0 & 2c-4a & c-a \end{vmatrix} = 0$$

$\Rightarrow (3b - 4a)(c - a) - (2c - 4a)(b - a) = 0$

$\Rightarrow 3bc - 3ab - 4ac + 4a^2$
$- (2bc - 2ca - 4ab + 4a^2) = 0$

$\Rightarrow bc + ab - 2ac = 0$

$\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \quad \Rightarrow$ a, b, c are in H.P.

PASSAGE 10 :

1. **(b) :** Since A is invertible

$\therefore \ AA^{-1} = A^{-1}A = I$

$\therefore \ |AA^{-1}| = |A^{-1}A| = |I| = 1$

$\Rightarrow \ |A^{-1}| = \frac{1}{|A|}$ *i.e.*, $\det(A^{-1}) = \frac{1}{\det A}$

2. **(b) :** Since $AB = AC$

$\therefore \ A^{-1}(AB) = A^{-1}(AC)$ if A^{-1} exists

$\Rightarrow (A^{-1}A)B = (A^{-1}A)C$

$\Rightarrow IB = IC \ \Rightarrow B = C$

Thus A is non-singular if $AB = AC \Rightarrow B = C$

3. **(d) :** Since $A^2 - A + I = O$

$\therefore \ A^{-1}A^2 - A^{-1}A + A^{-1}I = A^{-1} \cdot O = O$

$\Rightarrow A - I + A^{-1} = 0 \quad (\because \ A^{-1}A = I)$

$\Rightarrow A^{-1} = I - A$

Matrix Match Type

1. **(A) → (q), (B) → (s), (C) → (p)**

(A) Since, A satisfies the equation

$x^3 - 5x^2 + 4x + \lambda = 0$

$\Rightarrow A^3 - 5A^2 + 4A + \lambda I = O$

$\Rightarrow A(-A^2 + 5A - 4I) = \lambda I$

$\Rightarrow A\left\{\frac{1}{\lambda}(-A^2 + 5A - 4I)\right\} = I$, if $\lambda \ne 0$

Hence, A^{-1} exists and is equal to

$\frac{1}{\lambda}(-A^2 + 5A - 4I)$, if $\lambda \ne 0$.

(B) We have, $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \Rightarrow A^n = \begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix}$

[using the Principle of Mathematical Induction]

$$\Rightarrow \frac{1}{n}A^n = \begin{bmatrix} \frac{1}{n} & a \\ 0 & \frac{1}{n} \end{bmatrix} \Rightarrow \lim_{n\to\infty}\frac{1}{n}A^n = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

$\therefore \ \lambda = 1.$

(C) The given system of equations may be rewritten as $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 12 \end{bmatrix}$

Applying $R_2 \to R_2 - R_1, \ R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 6 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

For, $\lambda = 3$, we observe that rank of matrix A is 2 and that of the argumented matrix is 3. So, the system is inconsistent.

2. (A) → (q), (B) → (s), (C) → (p), (D) → (t), (E) → (r)

(A) $\therefore \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = 8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} + k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

or $\begin{bmatrix} 1+0 & 0+0 \\ -1-7 & 0+49 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

$= \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$

$\Rightarrow\ 1 = 8 + k,\ 49 = 56 + k\ \Rightarrow\ k = -7.$

(B) $\therefore \begin{vmatrix} \lambda & 7 & -2 \\ 4 & 1 & 3 \\ 2 & -1 & 2 \end{vmatrix} = 0\ \Rightarrow\ \lambda = \dfrac{2}{5}$

(C) $|\text{adj}\, A| = |A|^2$, where $|A| = 1$.

(D) Now, $\begin{vmatrix} -5 & 3 & 2 \\ 3 & 2 & -5 \\ 4 & -1 & -3 \end{vmatrix} = 0$ and $\begin{vmatrix} -5 & 3 \\ 3 & 2 \end{vmatrix} \neq 0.$

So, the only possible third-order minor is 0, while one second-order minor is non-zero. So, the rank is 2.

(E) Now, $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$ So, the rank is 3.

3. (A) → (p), (B) → (r), (C) → (q), (D) → (s)

(A) $A\cdot\text{adj}\ A = |A|\ I$ ... (1)

Taking determinants, $|\text{adj}\ A| = |A|^2$

(B) $\because\ |A|I = A(\text{adj}\ A)$

$\Rightarrow\ |A|I(\text{adj})^{-1} = A(\text{adj})(\text{adj})^{-1}$

$\Rightarrow\ |A|(\text{adj}\ A)^{-1} = AI \Rightarrow (\text{adj}\ A)^{-1} = \dfrac{A}{|A|}$

(C) $\text{adj}\ A \cdot (\text{adj adj}\ A) = |\text{adj}\ A|\ I$

$= |A|^2\ I$ ... (2)

Taking the determinants, $|\text{adj adj}\ A| = |A|^4$

(D) $\Rightarrow (\text{adj adj}\ A)^{-1} = \dfrac{A^{-1}}{|A|}$.

4. (A) → (p, s), (B) → (q), (C) → (r)

(A) s : $x + \lambda y = 1$ ... (1)

$\lambda x + y = \lambda^2$ ... (2)

(2) – λ (1) gives

$(1 - \lambda^2)\ y = \lambda\ (\lambda - 1)$

(A) $1 - \lambda^2 \neq 0 \Rightarrow \lambda \neq \pm 1$

(B) $\lambda = -1 \Rightarrow 0.y = 2 \Rightarrow$ no solution

(C) $\lambda = 1 \Rightarrow 0.y = 0 \Rightarrow$ infinity of solutions.

5. (A) → (q), (B) → (p)

(A) We have $f(0) = f(1) = 0$, therefore

$$\Delta_1 = \begin{vmatrix} f(0) & f(1) & f(2) \\ f(1) & f(2) & f(3) \\ f(2) & f(3) & f(4) \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 \\ 0 & 2 & 6 \\ 2 & 6 & 12 \end{vmatrix}$$

$= -8 = -3! - 2!$

Since $f'''(x) = f^{\text{iv}}(x) = 0\ \forall\ x \in R$

(B)

$$\Delta_2 = \frac{1}{4}\begin{vmatrix} f(a) & f'(a) & f''(a) \\ f'(a) & f''(a) & f'''(a) \\ f''(a) & f'''(a) & f^{iv}(a) \end{vmatrix} = \frac{1}{4}\begin{vmatrix} 2a(a-1) & 2a-1 & 2 \\ 2a-1 & 2 & 0 \\ 2 & 0 & 0 \end{vmatrix}$$

$= -2 = -2!$

6. (A) → (r), (B) → (s)

(A) Using ${}^{n+1}C_r = {}^{n}C_r + {}^{n}C_{r-1}$ in C_2 and C_3 and applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$.

We obtain $\Delta_1 = \begin{vmatrix} {}^{p+2}C_2 & {}^{p+2}C_1 & {}^{p+3}C_1 \\ {}^{p+3}C_2 & {}^{p+3}C_1 & {}^{p+4}C_1 \\ {}^{p+4}C_2 & {}^{p+4}C_1 & {}^{p+5}C_1 \end{vmatrix}$

Applying $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$ we obtain

$$\Delta_1 = \begin{vmatrix} {}^{p+2}C_2 & {}^{p+2}C_1 & {}^{p+3}C_1 \\ {}^{p+2}C_1 & 1 & 1 \\ {}^{p+3}C_1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} {}^{p+2}C_2 & p+2 & 1 \\ p+2 & 1 & 0 \\ p+3 & 1 & 0 \end{vmatrix}$$

using $C_3 \to C_3 - C_2$ we obtain

$= p + 2 - p - 3 = -1.$

(B) Applying $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$ we obtain

$$\Delta_2 = \begin{vmatrix} 1 & {}^{p}C_1 & {}^{p}C_2 \\ 0 & 1 & {}^{p}C_1 \\ 0 & 1 & {}^{p+1}C_1 \end{vmatrix} = \begin{vmatrix} 1 & p \\ 1 & p+1 \end{vmatrix} = 1.$$

7. (A) → (q), (B) → (r)

(A) Using $C_2 \to C_2 - C_1$ and

$C_3 \to C_3 - C_1$ we get

$$f'(x) = \begin{vmatrix} 2ax & -1 & b+1 \\ b & 1 & -b-1 \\ 2(ax+b) & 1 & -b \end{vmatrix} = \begin{vmatrix} 2ax+b & 0 & 0 \\ b & 1 & -b-1 \\ 2(ax+b) & 1 & -b \end{vmatrix}$$

[using $R_1 \to R_1 + R_2$]

$= 2ax + b$

As f is a maximum at $x = 5/2$, $f'(5/2) = 0$, therefore $5a + b = 0$. Integrating $f'(x)$ we obtain $f(x) = ax^2 + bx + c$.

As $f(0) = 2$, we get $c = 2$

since $f(1) = 1$, we get $1 = a + b + c$

$\Rightarrow a + b = 1$

Thus, $a = 1/4$, $b = -5/4$.

8. (A) → (q), (B) → (p)

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & \alpha \\ 4 & 7 & \beta \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & \alpha-6 \\ 0 & -1 & \beta-12 \end{vmatrix} = 6 - \alpha$$

[using $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$]

$|A| = 0 \Rightarrow \alpha = 6$, $|A| \neq 0 \Rightarrow \alpha \neq 6$.

9. (A) → (q), (B) → (s), (C) → (p)

We have

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & \lambda \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & \lambda-3 \end{vmatrix} = \lambda - 5.$$

If $|A| \neq 0$, the system has a unique solution. When $\lambda = 5$,

$$\text{Adj. } A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\text{and } (\text{Adj. } A)B = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ \mu \end{bmatrix} = \begin{bmatrix} -9 & +\mu \\ 18 & -2\mu \\ -9 & +\mu \end{bmatrix}.$$

Now, $(\text{Adj. } A)B = O$ if $\mu = 9$ and $(\text{Adj. } A)B \neq 0$ if $\mu \neq 9$.

Thus, the system has no solution if $\lambda = 5$ and $\mu \neq 9$ and infinite number of solutions when $\lambda = 5$, $\mu = 9$.

10. (A) → (p), (B) → (q)

(A) $$[A(\alpha, \beta)]^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & e^{-\beta} \end{bmatrix} = A(-\alpha, -\beta)$$

(B) $$[A(\alpha, \beta)]' = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & e^{\beta} \end{bmatrix} = A(-\alpha, \beta)$$

11. (A) → (s), (B) → (r), (C) → (p), (D) → (q)

(A) Since A is orthogonal $\therefore AA' = I = A'A$

$\therefore |AA'| = |I| = 1$

$\Rightarrow |A||A'| = 1$

$\Rightarrow |A||A| = 1 \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

(B) Clearly (B) → (r)

(C) det.(adj(adj A)) = $|A|^4 = (14)^4$

$$\left[\because \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix} = (1+2) - 2(-1-4) - 1(1-2)\right.$$

$$\left. = 3 + 10 + 1 = 14 \right]$$

(D) $(AA' - A'A)' = (AA')' - (A'A)'$

$= (A')'A' - A'(A')' = AA' - A'A$

$\therefore AA' - A'A$ is symmetric.

Integer Answer Type

1. **(4)** Putting $x = y = z = 1$ on both sides, we get

$$\begin{vmatrix} 1+1 & 1 & 1 \\ 1 & 1+1 & 1 \\ 1 & 1 & 1+1 \end{vmatrix} = k(1 \cdot 1 \cdot 1)$$

$\Rightarrow 2(4-1) - 1(2-1) + 1(1-2) = k$

$\Rightarrow k = 4$

2. **(3)** Given equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & \lambda-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu-6 \end{bmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu-10 \end{bmatrix}$$

If $\lambda = 3$ and $\mu \neq 10$, then $\rho(A) \neq \rho[A : B]$ and hence the system has no solution.

3. **(5)** If A is symmetric then $A^T = A$

$$\therefore A^T = \begin{bmatrix} 4 & 2x-3 \\ x+2 & x+1 \end{bmatrix} = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$$

$\Rightarrow x + 2 = 2x - 3 \Rightarrow x = 5$

4. **(4)** Since matrix is singular, then their determinant is zero

$$\text{or } \begin{bmatrix} 1 & 3 & \lambda+2 \\ 2 & 4 & 8 \\ 3 & 5 & 10 \end{bmatrix} = 0$$

$\Rightarrow\ 1(40-40)-3(20-24)+(\lambda+2)(10-12)=0$
$\Rightarrow\ 2\lambda = 12-4=8 \ \Rightarrow\ \lambda=4$

5. **(0)** $\Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$

$= \begin{vmatrix} x+1 & x+2 & x+a \\ 1 & 1 & b-a \\ 2 & 2 & c-a \end{vmatrix}$ $(R_2 \to R_2 - R_1$ &

$= \begin{vmatrix} x+1 & 1 & a-1 \\ 1 & 0 & b-a-1 \\ 2 & 0 & c-a-2 \end{vmatrix}$ $(C_3 \to C_3 - C_1,$

$= -1\begin{vmatrix} 1 & b-a-1 \\ 2 & c-a-2 \end{vmatrix}$ which is independent of x

$\therefore$ highest power of x is 0

6. **(8)** Let $\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -3 \\ 2 & 5 & -\lambda \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 3 & -\lambda+2 \end{vmatrix}$

$= -\lambda + 8$ (using $R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$)
If equation has a unique solution, then
$-\lambda + 8 \neq 0 \ \Rightarrow\ \lambda \neq 8$

7. **(0)** Given det. can be written as

$\begin{vmatrix} 2\sin A\cos A & \sin(A+B) & \sin(A+C) \\ \sin(A+B) & 2\sin B\cos B & \sin(B+C) \\ \sin(A+C) & \sin(B+C) & 2\sin C\cos C \end{vmatrix}$

($\because \sin C = \sin(\pi-(A+B)) = \sin(A+B)$, etc.)

$= \begin{vmatrix} \sin A & \cos A & 0 \\ \sin B & \cos B & 0 \\ \sin C & \cos C & 0 \end{vmatrix} \times \begin{vmatrix} \cos A & \cos B & \cos C \\ \sin A & \sin B & \sin C \\ 0 & 0 & 0 \end{vmatrix} = 0$

8. **(4)** $|A(\text{adj}A)| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$

$\Rightarrow\ |A||\text{adj}A| = 8$
$\Rightarrow\ |A||A|^2 = 8 \ \Rightarrow\ |A|^3 = 8 \ \Rightarrow\ |A| = 2$
($\because |\text{adj}A| = |A|^{n-1}$ if A is of order n)
$\therefore\ |\text{adj}A| = |A|^{3-1} = 2^2 = 4$

9. **(0)** $\Delta = \begin{vmatrix} 0 & i-1000 & i-500 \\ 1000-i & 0 & 1000-i \\ 500-i & i-1000 & 0 \end{vmatrix}$

Since the determinant is a skew-symmetric determinant of third order ($\Delta' = -\Delta$) and hence it is zero.

10. **(6)** Since A is a unit matrix
$\therefore\ A^2 = A$
$\therefore\ A^2 + 5A = A + 5A = 6A$
$\therefore\ n$ is 6.

11. **(4)** We know that in a square matrix of order n
$|\text{adj}\, A| = |A|^{n-1}$
$\Rightarrow\ |\text{adj}\,(\text{adj}\, A)| = |\text{adj}\, A|^{n-1} = |A|^{(n-1)^2}$

$|A|^{(n-1)^2} = |A|^9 \ \Rightarrow\ (n-1)^2 = 9$
$\therefore\ n^2 - 2n - 8 = 0 \ \Rightarrow\ n = 4$ as $n = -2$ is not possible.
Hence, value of $n = 4$.

12. **(5)** $D = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 2 & 5 & -2 \end{vmatrix} = 45,\ D_{1/x} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -4 & -2 \\ 3 & 5 & -2 \end{vmatrix} = 36,$

$D_{1/y} = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & -2 \\ 2 & 3 & -2 \end{vmatrix} = 9,\ D_{1/z} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -4 & 2 \\ 2 & 5 & 3 \end{vmatrix} = -9$

$x = \frac{5}{4}, y = 5, z = -5$
Hence, value of y is 5.

13. **(0)** $\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} x & y & z \\ 2(2^n-1) & 3(3^n-1) & 4(4^n-1) \\ 2(2^n-1) & 3(3^n-1) & 4(4^n-1) \end{vmatrix}$

$\Rightarrow\ \sum_{r=1}^{n} \Delta_r = 0$

14. **(4)** $\Delta = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 6 & 24 \\ 6 & 24 & 120 \end{vmatrix}$

Operate $R_3 \to R_3 - 6R_1, R_2 \to R_2 - 2R_1$, we get

$\Delta = \begin{vmatrix} 1 & 2 & 6 \\ 0 & 2 & 12 \\ 0 & 12 & 84 \end{vmatrix} = 168 - 144 = 24 = 4!$

15. **(0)** The given determinant can be rewritten as

$\begin{vmatrix} a_1^2 - 2a_1b_1 + b_1^2 & a_1^2 - 2a_1b_1 + b_2^2 & a_1^2 - 2a_1b_3 + b_3^2 & a_1^2 - 2a_1b_4 + b_4^2 \\ a_2^2 - 2a_2b_1 + b_1^2 & a_2^2 - 2a_2b_2 + b_2^2 & a_2^2 - 2a_2b_3 + b_3^2 & a_2^2 - 2a_2b_4 + b_4^2 \\ a_3^2 - 2a_3b_1 + b_1^2 & a_3^2 - 2a_3b_2 + b_2^2 & a_3^2 - 2a_3b_3 + b_3^2 & a_3^2 - 2a_3b_4 + b_4^2 \\ a_4^2 - 2a_4b_1 + b_1^2 & a_4^2 - 2a_4b_2 + b_2^2 & a_4^2 - 2a_4b_3 + b_3^2 & a_4^2 - 2a_4b_4 + b_4^2 \end{vmatrix}$

$= \begin{vmatrix} a_1^2 & -2a_1 & 1 & 0 \\ a_2^2 & -2a_2 & 1 & 0 \\ a_3^2 & -2a_3 & 1 & 0 \\ a_4^2 & -2a_4 & 1 & 0 \end{vmatrix} \times \begin{vmatrix} 1 & b_1 & b_1^2 & 0 \\ 1 & b_2 & b_2^2 & 0 \\ 1 & b_3 & b_3^2 & 0 \\ 1 & b_4 & b_4^2 & 0 \end{vmatrix}$

$= 0 \times 0 = 0$

16. **(6)** $BC = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \Rightarrow BC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$tr(A) + tr\left(\frac{A}{2}\right) + tr\left(\frac{A}{2^2}\right) + \ldots$

$= tr(A) + \frac{1}{2}tr(A) + \frac{1}{2^2}tr(A) + \ldots$

$= \frac{tr(A)}{1 - \left(\frac{1}{2}\right)} = 2\, tr(A) = 2(2+1) = 6$

17. **(0)** We know that a square matrix A is nilpotent of index n, if n is the least positive integer such that $A^n = O$ (null matrix).

For the given matrix, we have

$$A^2 = 0$$

Hence, it is a nilpotent matrix of index 2.

18. **(1)** The given matrix is singular.

$\therefore \begin{vmatrix} 3 & -1+x & 2 \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 3 & -1+x & 2 \\ 0 & -x & x \\ x & -x & 0 \end{vmatrix} = 0$

(Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$\Rightarrow 3x^2 + (1-x)(-x)^2 + 2(x^2) = 0$

Expanding along R_1

$\Rightarrow 4x^2 + x^3 = 0$

$\Rightarrow x^2(x+4) = 0 \Rightarrow x = 0, -4$

Clearly, $x = -4 \in [-4, -1]$

Hence, there is only one value of x in $[-4, -1]$ for which the given matrix is singular.

19. **(0)** Let A, B and C be three numbers such that $\alpha = B - C$, $\beta = C - A$ and $\gamma = A - B$

Clearly, $\alpha + \beta + \gamma = 0$

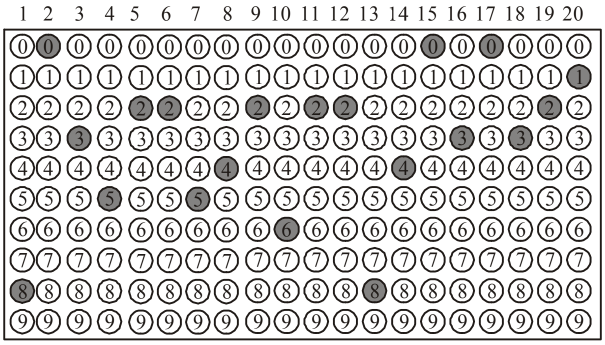
$$\Delta = \begin{vmatrix} \cos^2 A + \sin^2 A & \cos A\cos B + \sin A \sin B & \cos A\cos C + \sin A \sin C \\ \cos A\cos B + \sin A \sin B & \cos^2 B + \sin^2 B & \cos B\cos C + \sin B\sin C \\ \cos C\cos A + \sin C \sin A & \cos B \cos C + \sin B \sin C & \cos^2 C + \sin^2 C \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix}\begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} = 0 \times 0 = 0$$

20. **(9)** We have $|A| = \begin{vmatrix} \alpha & 2 \\ 2 & \alpha \end{vmatrix} = \alpha^2 - 4$

Now, $|A^3| = 125 \Rightarrow |A|^3 = 125$

$\Rightarrow (\alpha^2 - 4)^3 = 125 \Rightarrow \alpha^2 - 4 = 5 \Rightarrow \alpha^2 = 9$



PREVIOUS YEARS IIT-JEE/JEE Advanced

1. **(b) :** Let $\Delta = \begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} \sin x + 2\cos x & \cos x & \cos x \\ \sin x + 2\cos x & \sin x & \cos x \\ \sin x + 2\cos x & \cos x & \sin x \end{vmatrix}$$

$$= (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \sin x + 2\cos x\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix}$$

$= (\sin x + 2\cos x)(\sin x - \cos x)^2$

thus, $\Delta = 0$

$\Rightarrow (\sin x + 2\cos x)(\sin x - \cos x)^2 = 0$

$\Rightarrow \tan x = -2$ or $\tan x = 1$

As $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$, $\tan x = 1 \quad \therefore x = \frac{\pi}{4}$.

2. **(d) :** Given that $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

and $A^2 = B \Rightarrow \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \Rightarrow \alpha^2 = 1, \alpha + 1 = 5$

$\Rightarrow \alpha = \pm 1, \alpha = 4.$

$\because$ There is no common value.

3. **(a) :** The given system is, $x + ay = 0$, $az + y = 0$, $ax + z = 0$

It is the system of homogeneous equations, determinant of coefficient matrix is zero *i.e.*

$$\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0 \Rightarrow 1(1-0) - a(0 - a^2) = 0$$

$\Rightarrow 1 + a^3 = 0 \Rightarrow a^3 = -1 \Rightarrow a = -1.$

4. **(b) :** The coefficient determinant.

$$=\begin{vmatrix} 2 & -1 & +2 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = -3\lambda - 6$$

For no solution the necessary condition is $-3\lambda + 3 = 0 \Rightarrow \lambda = 1$

It can be seen that for $\lambda = 1$, there is no solution for the given system of equations.

5. **(c) :** $|A^3| = 125 \Rightarrow |A| = 5 \quad (\because \ |A^n| = |A|^n)$

$\Rightarrow \alpha^2 - 4 = 5 \Rightarrow \alpha = \pm 3$

6. **(b) :** $6I = A^3 + cA^2 + dA$

$$\text{Now } A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & 38 & 46 \end{bmatrix}$$

$$A^3 + cA^2 + dA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{bmatrix}$$

$$\ldots + \begin{bmatrix} c & 0 & 0 \\ 0 & -c & 5c \\ 0 & -10c & 14c \end{bmatrix} + \begin{bmatrix} d & 0 & 0 \\ 0 & d & d \\ 0 & -2d & 4d \end{bmatrix} = 6I$$

$\Rightarrow 1 + c + d = 6, -11 - c + d = 6,$

$19 + 5c + d = 0$

$-38 - 10c - 2d = 0, 46 + 14c + 4d = 6$

$c + d = 5, -c + d = 17, 5c + d = -19,$

$5c + d = -19, 7c + 2d = -20$

$\Rightarrow c = -6, d = 11$

7. **(b) :** Given $Q = PAP^T$

$Q^2 = QQ = PA\,(P^TP)\,AP^T = PA(IA)P^T = PA^2P^T$

Proceding in this way $Q^{2005} = PA^{2005}\,P^T$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \forall\, n \in N \quad (P^TP = PP^T = I)$$

$$Q^{2005} = P\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}P^T. \quad \text{Since } PP^T = I$$

$$\therefore\ P^T(Q^{2005})P = P^TP\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}P^TP$$

$$= I\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}I = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

8. **(a) : 1st solution :** The solution of $AX = b$ has only three possibilities.

(1) $\rho[A\,|\,b] \neq \rho(A)$, in which case the system is inconsistent and has no solution.

(2) $\rho[A\,|\,b] = \rho(A) = r$: consistent.

$r < n$ — Infinite solutions; $r = n$ — Unique solution

Thus the system $Ax = b$ can never have exactly two distinct solutions.

2nd Solution : A system of three non-homogeneous set of equations is nothing but three planes. And the determination of solution boils down to determining the points of intersection of three planes. And clearly three planes cannot intersect at two distinct points exactly.

9. **(c, d) :** $X^T = -X, Y^T = -Y, Z^T = Z$

$(Y^3Z^4 - Z^4Y^3)^T = (Y^3Z^4)^T - (Z^4Y^3)^T$

$= -Z^4Y^3 + Y^3Z^4 = Y^3Z^4 - Z^4Y^3$

Hence, symmetric.

Similarly $(X^{44} + Y^{44})^T = X^{44} + Y^{44}$, so symmetric.

$(X^4Z^3 - Z^3X^4)^T = (X^4Z^3)^T - (Z^3X^4)^T$

$= Z^3X^4 - X^4Z^3 = -(X^4Z^3 - Z^3X^4)$

Hence, skew symmetric.

Also, $(X^{23} + Y^{23})^T = -X^{23} - Y^{23}$

$= -(X^{23} + Y^{23})$, skew symmetric.

10. **(b, c) : 1st solution**

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$$

Applying $R_3 \to R_3 - R_2, R_2 \to R_2 - R_1$, we get

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 5+2\alpha & 5+4\alpha & 5+6\alpha \end{vmatrix} = -648\alpha$$

Applying $R_3 \to R_3 - R_2$, we get

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 2 & 2 & 2 \end{vmatrix} = -648\alpha$$

Applying $C_3 \to C_3 - C_2, C_2 \to C_2 - C_1$, we get

$$\begin{vmatrix} (1+\alpha)^2 & \alpha(2+3\alpha) & \alpha(2+5\alpha) \\ 3+2\alpha & 2\alpha & 2\alpha \\ 2 & 0 & 0 \end{vmatrix} = -648\alpha$$

Expanding, $2\alpha^2(3\alpha + 2) - 2\alpha^2(5\alpha + 2) = -324\alpha$

$\Rightarrow -4\alpha^3 = -324\alpha \Rightarrow \alpha(\alpha^2 - 81) = 0$

$\therefore\ \alpha = 0, -9, 9$

2nd solution

Writing the expression as a product of determinants, we have

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 4 & 2\alpha & \alpha^2 \\ 9 & 3\alpha & \alpha^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = -648\alpha$$

$$\Rightarrow \quad \alpha^3 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = -648\alpha$$

$\Rightarrow \alpha^3(-2)(4) = -648\alpha \quad \Rightarrow \alpha^3 = 81\alpha$

$\therefore \quad \alpha = 0, 9, -9$, as before

11. (a) : In the usual notation

$$D = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 0$$

$$D_1 = \begin{vmatrix} -1 & -2 & 3 \\ k & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 3 - k$$

$$D_2 = \begin{vmatrix} 1 & -1 & 3 \\ -1 & k & -2 \\ 1 & 1 & 4 \end{vmatrix} = k - 3$$

$$D_3 = \begin{vmatrix} 1 & -2 & -1 \\ -1 & 1 & -k \\ 1 & -3 & 1 \end{vmatrix} = k - 3$$

$D_1, D_2, D_3 = 0$ if $k = 3$

Thus the system of equation has no solution if k is different from 3, i.e. $k \neq 3$

Both the statement-1 and statement-2 are true and statement-2 also explains statement-1. The answer then is (a).

12. (a) : Let U_1 be

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ so that } \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Similarly $U_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$, $U_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$

Hence $U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix}$ and $|U| = 3$.

13. (b) : Moreover adj $U = \begin{bmatrix} -1 & -2 & 0 \\ -7 & -5 & -3 \\ 9 & 6 & 3 \end{bmatrix}$

Hence $U^{-1} = \dfrac{\text{adj } U}{3}$ and sum of the elements of $U^{-1} = 0$.

14. (a) : The value of $\begin{bmatrix} 3 & 2 & 0 \end{bmatrix} U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 4 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = -3 + 8 = 5.$$

15. (a) : There are two possibilities.

(1) Either all the diagonal elements are 1.
The number of matrices = ${}^3C_1 = 3$

(2) If two diagonal elements are zero and one element is 1.

The number of matrices = ${}^3C_1 \cdot {}^3C_1 = 9$

The number of all matrices = 3 + 9 = 12

16. (b) : For unique solution det $A \neq 0$

Case I : Let $A = \begin{bmatrix} 1 & h & g \\ h & 1 & f \\ g & f & 1 \end{bmatrix}$

det $A = 1 + 2fgh - f^2 - g^2 - h^2$

Also f, g, h are to be selected from 1, 0, 0

In this case det $A \neq 0$ for any values of f, g and h.

Case II : Let $A = \begin{bmatrix} 1 & h & g \\ h & 0 & f \\ g & f & 0 \end{bmatrix}$

det $A = 2fgh - f^2 \neq 0$

(... f, g, h are to be selected from 1, 0, 0)

This gives two cases.

Similarly we get 4 more cases.

Thus there are 6 matrices that give rise to unique solution.

17. (b) : Out of those matrices that have det = 0, we have to find out inconsistent system. There are exactly 4 matrices for which the system becomes inconsistent.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

18. (d) : $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$

Let A be symmetric matrix then $b = c$

We have det $A = a^2 - b^2$

As $p \mid a^2 - b^2$ we have $p \mid (a - b)(a + b)$

$p \mid a - b \Rightarrow a = b$ then p cases are possible

$p \mid a + b \Rightarrow a + b = p$ and so $(p - 1)$ case are possible

we have the total number of case $= p + p - 1$

$= 2p - 1$

Let A be skew symmetric $\Rightarrow a = 0$, now $b + c = 0$

$\det A = b^2$

But b^2 is never divisible by p, thus no matrices are possible in this case.

19. **(c) :** Trace $A = 2a$

$2a$ is divisible by p once if $a = 0$

Now $a \neq 0$, $\det A = a^2 - bc$

Each of have c is less than p at $b = p - r$ and $c = p - k$

$\det A = a^2 - (p - r)(p - k)$

$\det A$ is divisible by p. Thus $a^2 = rk$

Each of r and k can be selected in $p - 1$ ways.

Hence the number of matrices $= (p - 1)^2$

20. **(d) :** The desired number of matrices

$= p^3 - (a \neq 0$ and $\det A$ is divisible $p) - (a = 0$ and $\det A$ is divisible by $p)$

$= p^3 - (p - 1)^2 - (2p - 1) = p^3 - p^2$

21. **(3)** View the equation in xyz, y and t.

We have

$(xyz)\sin3\theta - y\cos3\theta - z\cos3\theta = 0$

$(xyz)\sin3\theta - 2y\sin3\theta - 2z\cos3\theta = 0$

$(xyz)\sin3\theta - y(\cos3\theta + \sin3\theta) - 2z\cos3\theta = 0$

$xyz \neq 0$

Hence the equation has non-trivial solution which gives

$$\begin{vmatrix} \sin3\theta & -\cos3\theta & -\cos3\theta \\ \sin3\theta & -2\sin3\theta & -2\cos3\theta \\ \sin3\theta & -(\cos3\theta + \sin3\theta) & -2\cos3\theta \end{vmatrix} = 0$$

$\Rightarrow \sin3\theta\cos3\theta(\sin3\theta - \cos3\theta) = 0$

$\Rightarrow \sin3\theta = 0$ then $xyz = 0$ (not possible)

$\cos3\theta = 0$ not possible

$\sin3\theta = \cos3\theta \Rightarrow \tan3\theta = 1$

$$3\theta = n\pi + \frac{\pi}{4}, n \in z$$

$$\theta = \frac{n\pi}{3} + \frac{\pi}{12} \; ; \; \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}$$

Thus there are 3 solutions.

22. **(0)** $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} z & z & z \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$\Rightarrow z\begin{vmatrix} 1 & 1 & 1 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$\Rightarrow z\{(z + \omega^2)(z + \omega) - 1 - \omega(z + \omega - 1) + \omega^2(1 - z - \omega^2)\} = 0$

which gives $z^3 = 0$

Thus $z = 0$ is the only distinct solution.

23. **(4)** $\det A = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$

$$= \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ 0 & (2k+1) & -(2k+1) \end{vmatrix} \quad R_3 \to R_3 + R_2$$

$$= (2k+1)\begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ 0 & 1 & -1 \end{vmatrix}$$

$$= (2k+1)\begin{vmatrix} 2k-1 & 4\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1-2k & -2k \\ 0 & 0 & -1 \end{vmatrix} \quad C_2 \to C_2 + C_3$$

$= (2k + 1)^3$ (By expansion)

$\det B = 0$ ($\because B$ is skew symmetric matrix of odd order)

$\det(\text{adj } A) = (\det A)^{n-1} = \{(2k + 1)^3\}^2 = (2k + 1)^6$

Now $(2k + 1)^6 = 10^6 \Rightarrow 2k + 1 = 10 \Rightarrow 2k = 9$

Thus $[k] = 4$