JEE MAIN

Class XI

1 Sequences and Series

- **Arithmetic and Geometric progressions, insertion of arithmetic, geometric means between two given numbers, Relation between A.M. and G.M., Sum upto n terms of special series: S_n, S_{n^2}, S_{n^3}, Arithmetico-Geometric progression.**

ARITHMETIC PROGRESSION

- Sequence of numbers is said to be in A.P. when the difference between the consecutive terms (numbers) is always same. If a is the first term and d is the common difference, the A.P. can be written as a, $a + d$, $a + 2d$, ..., $a + (n - 1)d$.
  $a + (n - 1)d$ is called the **general term** denoted by $\boldsymbol{t_n}$.
  Sometimes we call the general term as the **last term (l)**.
  i.e., $t_n = a + (n - 1)d$.

Sum of n terms of an A.P.

- Let S_n be the sum of n terms of an A.P.
  $S_n = a + (a + d) + (a + 2d) + ... + (a + (n - 1)\ d)$ ...(1)
- Now, rewriting the terms in the reverse order, we get
  $S_n = a + (n - 1)\ d + a + (n - 2)\ d + ...a + d + a$ ...(2)
- Adding (1) and (2), we get $2S_n = [2a + (n - 1)\ d] + [2a + (n - 1)\ d] + ... + [2a + (n - 1)d]$ $= n\ [2a + (n - 1)d]$ (upto n terms)

  $\Rightarrow$ $\boldsymbol{S_n = \frac{n}{2}[a + a + (n - 1)d]}$

  where $a + (n - 1)d = l$ = last term of A.P.

  $\therefore$ $\boldsymbol{S_n = \frac{n}{2}\ [a + l]}$

Arithmetic Mean between two numbers

- If a, x, b are in A.P.

  Then the A.M. of a and b is $x = \frac{a+b}{2}$

Insertion of n Arithmetic Means between two numbers

- If n numbers are introduced between two numbers a, b such that a, A_1, A_2, ..., A_n, b forms an A.P., then A_1, A_2, ..., A_n are known as n A.M.'s between a and b.
- Let us suppose d is the common difference for this newly formed A.P.

  Thus $b = a + (n + 2 - 1)d$ $\Rightarrow$ $d = \frac{b-a}{n+1}$

  Thus, $A_1 = a + d = a + \frac{b-a}{n+1}$ [By definition of A.P.]

  $A_2 = a + 2d = a + 2\left[\frac{b-a}{n+1}\right]$

  Thus n^{th} A.M. $(A_n) = a + nd = a + n\left[\frac{b-a}{n+1}\right]$.

Special forms of A.P.

- If 3 numbers are in A.P. we may take them as $a - d$, a,$a + d$.
- If 4 numbers are in A.P. we can take them as $a - 3d$, $a - d$,$a + d$, $a + 3d$.

ILLUSTRATION

1 **There are four positive numbers in A.P. The sum of the squares of the extremes is 272 and the sum of the squares of the means is 208. Find the numbers ?**

Soln.: Let the four numbers $a - 3d$, $a - d$, $a + d$, $a + 3d$ be in A.P.

$(a - 3d)^2 + (a + 3d)^2 = 2(a^2 + 9d^2) = 272$

$(a - d)^2 + (a + d)^2 = 2(a^2 + d^2) = 208$

$\Rightarrow a^2 + 9d^2 = 136,\ a^2 + d^2 = 104$

$\Rightarrow d = 2,\ a = 10.$

The numbers are 4, 8, 12, 16.

2 **In the sequence 1, 3, 8, 16, 27, 41,....the difference between adjacent terms form an A.P. Find the n^{th} term of the sequence ?**

Soln.: $t_2 - t_1 = 2$

$t_3 - t_2 = 5$

$t_4 - t_3 = 8$

$t_{n+1} - t_n = n^{th}$ terms of the A.P. 2, 5, 8,... $= 2 + (n - 1)3 = 3n - 1$

$(t_2 - t_1) + (t_3 - t_2) + \ldots.. + (t_{n+1} - t_n) = 2 + 5 + 8 + \ldots + (3n - 1) = \frac{n}{2}(3n + 1)$

$\therefore\ t_{n+1} - t_1 = \dfrac{3n^2 + n}{2}$

$t_{n+1} = 1 + \dfrac{3n^2 + n}{2} = \dfrac{3n^2 + n + 2}{2}$ $\quad (\because\ t_1 = 1)$

Replace $n \to n - 1$

$\therefore\ t_n = \dfrac{3(n-1)^2 + (n-1) + 2}{2} = \dfrac{3n^2 - 5n + 4}{2}.$

3 **If the sum of m terms of an A.P. is equal to the sum of either next n terms or the next p terms, then find the value of $\dfrac{\frac{1}{m} - \frac{1}{n}}{\frac{1}{m} - \frac{1}{p}}$?**

Soln.: $S_{m+n} = S_{m+p} = S_m + S_m = 2S_m$

$S_{m+n} = 2S_m$

$\Rightarrow \dfrac{(m+n)}{2}\left[2a + (m + n - 1)d\right] = m[2a + (m - 1)d]$

$(n - m)a = \left[m^2 - m - \dfrac{(m+n)^2}{2} + \dfrac{m+n}{2}\right]d$

$\Rightarrow \dfrac{2a}{d} = 1 - m - n - \dfrac{2mn}{n - m}$ ...(1)

Likewise, $S_{m+p} = 2S_m \Rightarrow \dfrac{2a}{d} = 1 - m - p - \dfrac{2mp}{p - m}$ ...(2)

(1), (2) $\Rightarrow n + \dfrac{2mn}{n - m} = p + \dfrac{2mp}{p - m}$

$\Rightarrow \dfrac{n(m+n)}{n - m} = \dfrac{p(m+p)}{p - m}$

$n(m + n)(p - m) = p(m + p)(n - m)$

Dividing both sides by mnp, we get

$(m + n)\left(\dfrac{1}{m} - \dfrac{1}{p}\right) = (m + p)\left(\dfrac{1}{m} - \dfrac{1}{n}\right) \quad \therefore\ \dfrac{\frac{1}{m} - \frac{1}{n}}{\frac{1}{m} - \frac{1}{p}} = \dfrac{m + n}{m + p}$

4 **If n arithmetic means are inserted between 1 and 31 such that the ratio of the first mean and n^{th} mean is 3 : 29, find the value of n.**

Soln.: First and last term of the A.P. are 1 and 31.

Also n arithmetic means $A_1, A_2, \ldots, A_n$ are inserted in between them resulting in a A.P. of $n + 2$ terms.

Given that $A_1 : A_n = 3 : 29$

$d = \frac{b-a}{n+1}$, where b is last term, a is first term and $A_r = a + r\left[\frac{b-a}{n+1}\right]$

$A_1 = a + 1\left[\frac{b-a}{n+1}\right]$ where $b = 3$, $a = 1$

$\therefore \quad A_1 = 1 + \frac{30}{n+1} = \frac{n+31}{n+1}$

Similarly, $A_n = 1 + \frac{n(30)}{n+1} = \frac{n+1+30n}{n+1} = \frac{31n+1}{n+1}$

$\therefore \quad A_1 : A_n = \frac{n+31}{31n+1} = \frac{3}{29}$ (given) $\Rightarrow n = 14$.

5 **If the sum of n terms of two A.P.'s are in the ratio $(4n + 1) : (2n + 1)$, find the ratio of their 4^{th} terms.**

Soln.: Given two A.P.'s say (a_1, d_1) and (a_2, d_2), where (a_i, d_i) means A.P. with first term a_i and common difference d_i. Also ratio of $S_{1n} : S_{2n}$ is $(4n + 1) : (2n + 1)$

Sum of n terms of an A.P. $= \frac{n}{2}(2a + (n-1)d)$ and n^{th} term of an A.P. $= a + (n-1)d$.

Given that $S_{1n} : S_{2n} = \frac{4n+1}{2n+1} \Rightarrow \frac{\frac{n}{2}(2a_1 + (n-1)d_1)}{\frac{n}{2}(2a_2 + (n-1)d_2)} = \frac{4n+1}{2n+1}$

or $$\frac{a_1 + \frac{1}{2}(n-1)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{4n+1}{2n+1} \quad \ldots(1)$$

Also, ratio of their 4^{th} term is $\frac{a_1 + (4-1)d_1}{a_2 + (4-1)d_2} = \frac{a_1 + 3d_1}{a_2 + 3d_2}$ ...(2)

$\therefore \quad$ Comparing (1) and (2) we get $\frac{n-1}{2} = 3 \Rightarrow n - 1 = 6 \Rightarrow n = 7$

$\therefore \quad$ for $n = 7$ ratio will be $\frac{28+1}{14+1} = \frac{29}{15}$

6 **How many terms of the series 54, 51, 48, 45, ... must to be taken to make the sum 513.**

Soln.: We have A.P. = 54, 51, 48, ... where $a = 54$ and $d = 51 - 54 = -3$.

S_n (sum of n terms of progression) $= \frac{n}{2}(2a + (n-1)d)$.

$S_n = \frac{n}{2}[108 + (n-1)(-3)]$.

Also $S_n = 513$ (given)

$\Rightarrow \quad \frac{n}{2}[108 + (n-1)(-3)] = 513 \Rightarrow 108n - 3n^2 + 3n = 1026$

$\Rightarrow \quad 3n^2 - 111n + 1026 = 0$ or $n^2 - 37n + 342 = 0$

On solving we get $n = 18$ or 19

Now $T_{19} = a + (n-1)d = 54 + 18 \times (-3) = 0. \quad \therefore \quad n = 18$.

7 **The internal angles of a plane polygon are in A.P. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon and the sum of internal $\angle s$.**

Soln.: Given that internal angles of the polygon are in A.P., such that its first term is the smallest angle = 120° and d = 5°.

For a polygon of n sides sum of internal angles is $(n - 2) \times 180°$

Also sum of n terms of an A.P. is $S_n = \frac{n}{2}(2a + (n - 1)\,d)$

For a = 120°, d = 5°

$\frac{n}{2}\,(240 + (n - 1)\,5) = (n - 2)\,180°$ (given)

$\Rightarrow\ 120n + 5\frac{n^2}{2} - \frac{5n}{2} = 180n - 360 \Rightarrow n^2 - 25n + 144 = 0$ or $n = 9, 16$

Since interior angles are < 180° $\therefore$ T_{16} = 195° cannot be interior angles

So, $n = 9$ $\therefore$ Sum of interior angles = (9 – 1) 180° = 1440°

8 **If $a_i > 0$ (i = 1, 2, ..., n) be in A.P. Show that $\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots+\frac{1}{a_{n-1}a_n}=\frac{n-1}{a_1a_n}$.**

Soln.: Given that $a_i > 0$ such that $a_1, a_2, \ldots a_n$ forms an A.P.

For $a_1, a_2, \ldots, a_n$ to be in A.P., so $a_2 - a_1 = a_3 - a_2 = a_n - a_{n-1} = d$ where d is the common difference.

Consider $\frac{1}{a_1a_2} = \frac{d}{d(a_1a_2)} = \frac{a_2 - a_1}{d(a_1a_2)} = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_2}\right)$

Similarly, $\frac{1}{a_2a_3} = \frac{d}{d(a_2a_3)} = \frac{1}{d}\left[\frac{1}{a_2} - \frac{1}{a_3}\right]$

On the similar pattern we will get $\frac{1}{a_na_{n-1}} = \frac{1}{d}\left[\frac{1}{a_{n-1}} - \frac{1}{a_n}\right]$

$$\therefore\quad \frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots+\frac{1}{a_{n-1}a_n} = \frac{1}{d}\left[\frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_2}-\frac{1}{a_3}+\ldots+\frac{1}{a_{n-1}}-\frac{1}{a_n}\right] = \frac{1}{d}\left[\frac{1}{a_1}-\frac{1}{a_n}\right]$$

$$= \frac{1}{d}\,\frac{a_n - a_1}{a_1a_n} = \frac{a_1 + (n-1)d - a_1}{d\,a_1a_n} = \frac{n-1}{a_1a_n}.$$

GEOMETRIC PROGRESSION

- G.P. is a sequence whose first term is non zero & each of whose succeeding term is r times the preceding term, where r is some fixed non zero number, known as the common ratio of G.P. If a is the first term of a G.P. with common ratio r then its n^{th} term t_n, is given by $\boldsymbol{t_n = ar^{n-1}}$.
- A Geometric Progression (G.P.) is a sequence of numbers, if the ratio of any term and its just preceding term is a constant throughout. This constant factor is known as common ratio of the G.P. The set of numbers $a_1, a_2, \ldots, a_n$ is said to be G.P. sequence if

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots = \frac{a_n}{a_{n-1}} = \text{constant } (r).$$

Sum of n terms of the G.P.

- Let S_n be the sum of n terms of the G.P. Then S_n is denoted by

$$\boldsymbol{S_n = \frac{a(r^n - 1)}{r - 1} \text{ or } \frac{a(1 - r^n)}{1 - r}} \text{ if } r \neq 1$$

(Depending on whether $r > 1$ or $r < 1$)

- For $r = 1$ the G.P. will become $a, a, a, ..., a \Rightarrow$ $\boldsymbol{S_n = na}$ for $r = 1$
- Sum of an infinite G.P. is given by $\dfrac{\boldsymbol{a}}{\boldsymbol{1-r}}$, where $|r| < 1$.

Geometric mean between two numbers

- If a, x, b are in G.P., then $x = \sqrt{ab}$ is the G.M. of a and b.

Insertion of *n* Geometric Means between two numbers

- Let a, b be two numbers. If n numbers $G_1, G_2, ..., G_n$ are introduced in between them such that $a, G_1, G_2, ..., G_n, b$ forms a G.P. then $G_1, G_2, ..., G_n$ are termed as n G.M's between a, b.
- Now if r is common ratio for this G.P. then

  $b = a\, r^{n+2-1} \Rightarrow r = \left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$

  Hence, $G_1 = a\left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$; $G_2 = a\left(\dfrac{b}{a}\right)^{\frac{2}{n+1}}$

  So n^{th} term will be $G_n = a\left(\dfrac{b}{a}\right)^{\frac{n}{n+1}}$.

Special Forms of G.P.

- If 3 numbers are in G .P., we may take them as $\dfrac{a}{r}, a, ar$.
- If 4 numbers are in G.P., we may take them as $\dfrac{a}{r^3}, \dfrac{a}{r}, ar, ar^3$.

ILLUSTRATION

9 **The sum of the squares of three distinct positive numbers, which are in G.P., is s^2. If their sum is αs, find the interval in which α^2 lies.**

Soln.: Let the numbers be $\dfrac{a}{r}, a, ar$ where $r \neq 1, -1, 0$ and $a > 0$.

$$a^2\left(\frac{1}{r^2}+1+r^2\right) = s^2,\ a\left(\frac{1}{r}+1+r\right) = \alpha s$$

$$\therefore \quad \alpha^2 = \frac{\left(\frac{1}{r}+1+r\right)^2}{\frac{1}{r^2}+1+r^2} = \frac{(1+r+r^2)^2}{1+r^2+r^4}$$

$$\because \quad \alpha^2 = \frac{1+r+r^2}{1-r+r^2}$$

$r^2(\alpha^2 - 1) - (\alpha^2 + 1)r + \alpha^2 - 1 = 0$

r is real $\Rightarrow$ Discriminant ≥ 0

$(\alpha^2 + 1)^2 - 4(\alpha^2 - 1)^2 \geq 0$

$(\alpha^2 + 1 + 2(\alpha^2 - 1))\ (\alpha^2 + 1 - 2(\alpha^2 - 1)) \geq 0$

$(3\alpha^2 - 1)\ (3 - \alpha^2) \geq 0$

$\Rightarrow \left(\alpha^2 - \dfrac{1}{3}\right)(\alpha^2 - 3) \leq 0 \Rightarrow \alpha^2 \in \left[\dfrac{1}{3}, 3\right]$. But $r \neq 0, \pm 1 \Rightarrow \alpha^2 \in \left(\dfrac{1}{3}, 1\right) \cup (1, 3)$.

10 **Let a, b, c, d, be in G.P. If u, v, w satisfy the system of equations $u + 2v + 3w = 6$, $4u + 5v + 6w = 12$, $6u + 9v = 4$, then show that the roots of the equations $\left(\dfrac{1}{u}+\dfrac{1}{v}+\dfrac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]x + u + v + w = 0$ and $20x^2 + 10(a-d)^2 x - 9 = 0$ are reciprocals of each other.**

Soln.: Solving the system of equations, we get $u = -\frac{1}{3}$, $v = \frac{2}{3}$, $w = \frac{5}{3}$

$$\therefore \quad \frac{1}{u}+\frac{1}{v}+\frac{1}{w} = -\frac{9}{10}, \; u + v + w = 2$$

$$(b - c)^2 + (c - a)^2 + (d - b)^2 = (a - d)^2$$

The quadratic equation $\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2 + [(b-c)^2+(c-a)^2+(d-b)^2]x + u + v + w = 0$ becomes $-\frac{9}{10}x^2 + (a - d)^2x + 2 = 0$ or $- 9x^2 + 10\,(a - d)^2\,x + 20 = 0$. The equation whose roots are the reciprocals of the roots of this equation is obtained by writing $\frac{1}{x}$ in place of x.

Thus $-\frac{9}{x^2}+10\frac{(a-d)^2}{x}+20 = 0$ or $20x^2 + 10\,(a - d)^2\,x - 9 = 0$.

11 **If S_1, S_2, S_3,....., S_n are the sums of n terms of n G.P.s for which all the first terms are 1 and common ratios are 1, 2, 3,...., n respectively. Find the value of $S_1 + \sum\limits_{r=2}^{n}(r-1)S_r$?**

Soln.: $S_r = 1 + r + r^2 + \ldots. + r^{n-1}$, $S_1 = n$, $S_r = \frac{1-r^n}{1-r}$, $r > 1$

$$\therefore \quad S_1 + \sum_{r=2}^{n}(r-1)S_r = n + \sum_{r=2}^{n}(r^n-1) = 1^n + 2^n + 3^n + \ldots.\; n^n.$$

12 **If p^{th}, q^{th} and r^{th} term of a G.P. are respectively x, y and z then prove that $x^{q-r} \cdot y^{r-p} \cdot z^{p-q} = 1$.**

Soln.: We have p^{th}, q^{th} and r^{th} terms of G.P. as x, y and z.

A G.P. with first term a and common ratio R has n^{th} term as $t_n = a\,R^{n-1}$.

Given p^{th} term is $x \Rightarrow a\,R^{p-1} = x$

Similarly, $a\,R^{q-1} = y$; $a\,R^{r-1} = z$

Now $x^{q-r} \cdot y^{r-p} \cdot z^{p-q} = a^{q-r} \cdot a^{r-p} \cdot a^{p-q} \cdot r^{(p-1)(q-r)} \cdot r^{(q-1)(r-p)} \cdot r^{(r-1)(p-q)} = a^0r^0 = 1$.

13 **The sum of first three terms of a G.P. is to the sum of first 6 terms as 125 : 152. Find the common ratio of the G.P.?**

Soln.: Given the ratio of sum of first three terms to the sum of first 6 terms is 125 : 152.

Sum of n terms of a G.P. with first term a and common ratio r is $S_n = a\frac{(1-r^n)}{1-r}$.

$$S_3 = a\frac{(1-r^3)}{1-r};\; S_6 = a\frac{(1-r^6)}{1-r}.$$

$S_3 : S_6 = 125 : 152$

$$\therefore \quad \frac{a(1-r^3)}{1-r}\times\frac{1-r}{a(1-r^6)} = \frac{125}{152} \Rightarrow \frac{1-r^3}{1-r^6} = \frac{125}{152}$$

$$\text{or} \quad \frac{(1-r^3)}{(1-r^3)(1+r^3)} = \frac{125}{152} \Rightarrow 152 = 125 + 125\,r^3 \Rightarrow \frac{27}{125} = r^3 \Rightarrow r = 3/5$$

(14) Find the sum upto n terms of the series: 0.7 + 0.77 + 0.777 + ... ?

Soln.: We will try to make either an A.P. or a G.P. of the given series so as we can apply the direct formula to find the sum.

$0.7 + 0.77 + 0.777 + \ldots$ upto n terms $= (0.7 + 0.77 + 0.777 + \ldots)\frac{9}{9} = \frac{7}{9}(0.9 + 0.99 + \ldots)$

$= \frac{7}{9}(1 - 0.1 + 1 - 0.01 + \ldots) = \frac{7}{9}[n - (0.1 + 0.01 + \ldots\ n \text{ terms})]$

$= \frac{7}{9}\left[n - \left(\frac{1-(1/10)^n}{1-1/10}\right)\frac{1}{10}\right] = \frac{7n}{9} - \frac{7}{81}\left(1 - \frac{1}{10^n}\right).$

Relation between A.M. and G.M

- **A.M. – G.M. Inequality:**

  $\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1\, a_2\, \ldots.\, a_n)^{1/n}$

  Equality holds if and only if

  $a_1 = a_2 = \ldots. = a_n$
- **Root mean square inequality:**

  $\sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \ldots + a_n}{n}$

Note

- If A and G are respectively arithmetic and geometric means between two positive quantities a and b, then the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$
- If A and G are the A.M. and G.M. between two positive numbers, then the numbers are $A \pm \sqrt{A^2 - G^2}$.

Sum upto *n* terms of special series

- **Series :** If $\{t_1, t_2, \ldots., t_n\}$ is a sequence, then $S = t_1 + t_2 + \ldots + t_n$ is called the corresponding series.
- **Summation of Series :** We can in general, find the sum of an A.P., G.P. and A.G.P. upto n terms.

  Now we will discuss about summation and its general properties.

  The Σ denotes summation. For instance we can write $a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i$
- **Some Basic Properties of Σ are :**

(i) $\sum_{r=1}^{n}(a_r + b_r) = \sum_{r=1}^{n} a_r + \sum_{r=1}^{n} b_r$ (ii) $\sum_{r=1}^{n} k \cdot a_r = k \cdot \sum_{r=1}^{n} a_r$

(iii) $\sum_{r=1}^{n} k = kn$ (iv) $\sum_{r=1}^{n} 1 = n$

(v) $\sum_{r=1}^{n} r = 1 + 2 + \ldots n = \frac{n(n+1)}{2}$ = (sum of first n natural numbers)

(vi) $\sum_{r=1}^{n} r^2 = 1 + 2^2 + 3^2 + \ldots n^2 = \frac{n(n+1)(2n+1)}{6}$

(vii) $\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2 = \left[\sum_{r=1}^{n} r\right]^2$

- **Summation of Series :** The problems involving summation of series can be divided into two steps.

  (i) Determination of n^{th} term T_n (ii) Finding the sum (S_n).

ARITHMETIC-GEOMETRIC PROGRESSION (A.G.P.)

- $a, (a + d)r, (a + 2d)r^2, (a + 3d)r^3, \ldots..$ be the terms of A.G.P. ....(1)

  $t_n = (a + (n - 1)d)r^{n-1}$

  (1) is A.P. if $r = 1$ and G.P. if $d = 0$

For example : 1, 3, 5, –25 is A.G.P.

where $a = 1,\ d = -\frac{2}{5},\ r = 5$

The sum of n terms of (1) is $S_n = \frac{1}{1-r}\left[a - (a + (n-1)d)r^n + \frac{dr}{1-r}\left(1 - r^{n-1}\right)\right]$

- Sum of infinite G.P. $a,\ ar^2,\ ar^3, ...$ is $S_\infty = \frac{a}{1-r}$
- Sum of infinite A.G.P. $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2} = \frac{a+(d-a)r}{(1-r)^2}$

ILLUSTRATION

15 Sum upto n terms: $\frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{5\cdot7\cdot9} + ...$ and hence find the sum upto infinity.

Soln.: Determination of n^{th} term:

$$T_n = \frac{1}{(2n-1)(2n+1)(2n+3)} = \frac{A}{(2n-1)} + \frac{B}{(2n+1)} + \frac{C}{(2n+3)}$$

$\Rightarrow 1 = A(2n + 1)(2n + 3) + B(2n - 1)(2n + 3) + C(2n - 1)(2n + 1)$

By comparing coefficient on both sides, we get $A = \frac{1}{8};\ B = -\frac{1}{4};\ C = \frac{1}{8}$

$$T_n = \frac{1}{8}\left\{\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right] - \left[\frac{1}{(2n+1)} - \frac{1}{2n+3}\right]\right\}$$

Summation of T_n : $S_n = \Sigma T_n$

$$T_1 = \frac{1}{8}\left\{\left(\frac{1}{1} - \frac{1}{3}\right) - \left(\frac{1}{3} - \frac{1}{5}\right)\right\};\ T_2 = \left(\frac{1}{8}\right)\left\{\left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{1}{5} - \frac{1}{7}\right)\right\}$$

$$T_3 = \left(\frac{1}{8}\right)\left\{\left(\frac{1}{5} - \frac{1}{7}\right) - \left(\frac{1}{7} - \frac{1}{9}\right)\right\};\ ......;\ T_n = \left(\frac{1}{8}\right)\left[\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) - \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)\right]$$

On addition we get; $S_n = T_1 + T_2 + ... + T_n = \frac{1}{8}\left\{1 - \frac{1}{3} - \frac{1}{2n+1} + \frac{1}{2n+3}\right\}$

$$\therefore\quad S_n = \frac{1}{8}\left\{\left(1 - \frac{1}{2n+1}\right) - \left(\frac{1}{3} - \frac{1}{2n+3}\right)\right\} \text{ or } S_n = \frac{1}{3}\frac{n(n+2)}{(2n+1)(2n+3)}$$

$\therefore$ Taking limit as $n \to \infty$; we get sum of infinite terms $S_\infty = \frac{1}{12}$.

16 Find the sum of n terms of the series $1\cdot3 + 3\cdot8 + 5\cdot15 + 7\cdot24 + ...$

Soln.: Determination of n^{th} term T_n:

The n^{th} term of sequence 1, 3, 5, ... is $2n - 1$.

The n^{th} term of sequence 3, 8, 15, ... is $n(n + 2)$.

Thus n^{th} term of the given series will be $(2n - 1)(n + 2)n = 2n^3 - 2n + 3n^2$

$\therefore\quad T_n = 2n^3 + 3n^2 - 2n$

Summation of series :

$S_n = \Sigma T_n = \Sigma 2n^3 + \Sigma 3n^2 - \Sigma 2n = 2\Sigma n^3 + 3\Sigma n^2 - 2\Sigma n$

$$S_n = 2.\frac{(n(n+1))^2}{4} + \frac{3n(n+1)(2n+1)}{6} - 2\frac{(n(n+1))}{2}$$

$$S_n = \frac{[n(n+1)]^2}{2} - [n(n+1)] + \frac{1}{2}(n+1)(2n+1)n$$

$$S_n = \frac{n(n+1)}{2}[n^2 + n - 2 + 2n + 1] = \frac{n(n+1)}{2}[n^2 + 3n - 1].$$

17 **Find the sum of n terms of the series $1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots$**

Soln.: Determination of n^{th} term : $T_n = \dfrac{1}{1+2+3+\ldots+n}$

$$T_n = \frac{1}{\frac{n(n+1)}{2}} = \frac{2}{n(n+1)} = 2\left[\frac{1}{n}-\frac{1}{n+1}\right] \text{ [using partial fractions]}$$

$$\therefore \quad T_n = 2\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

Summation of series: $S_n = \Sigma T_n$

$$T_1 = 2\left[1-\frac{1}{2}\right] \ ; \ T_2 = 2\left[\frac{1}{2}-\frac{1}{3}\right] \ ; \ T_3 = 2\left[\frac{1}{3}-\frac{1}{4}\right] \ ; \ \ldots\ldots \ ; \ T_n = 2\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

Adding them we get

$$S_n = T_1 + T_2 + \ldots. + T_n = 2\left[1-\frac{1}{n+1}\right] = 2\left[\frac{n+1-1}{n+1}\right] \quad \therefore \quad S_n = \left[\frac{2n}{n+1}\right].$$

18 **The real numbers x_1, x_2, x_3 satisfies the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P. Find the intervals in which β and γ lie.**

Soln.: $x_1 = x_2 - d$, x_2, $x_3 = x_2 + d$ are in A.P. and are the roots of the cubic equation.

$x_1 + x_2 + x_3 = 1$ ...(1)

$x_1 + x_2 + x_2 x_3 + x_3 x_1 = \beta$ ...(2)

$x_1 x_2 x_3 = -\gamma$ ...(3)

$$(1) \Rightarrow 3x_2 = 1 \ \Rightarrow x_2 = \frac{1}{3}$$

$$(2) \Rightarrow \beta = \left(\frac{1}{3}-d\right)\frac{1}{3}+\frac{1}{3}\left(\frac{1}{3}+d\right)+\left(\frac{1}{3}+d\right)\left(\frac{1}{3}-d\right) = \frac{1}{3}-d^2 \leq \frac{1}{3}$$

$$(3) \Rightarrow \gamma = -\left(\frac{1}{3}-d\right)\frac{1}{3}\left(\frac{1}{3}+d\right) = \frac{d^2}{3}-\frac{1}{27} \geq -\frac{1}{27}$$

$$\therefore \beta \in \left(-\infty, \frac{1}{3}\right], \gamma \in \left[-\frac{1}{27}, \infty\right).$$

19 **If a, b, c are positive, then prove that $[(1 + a)\,(1+ b)\,(1 + c)]^7 > 7^7\, a^4\, b^4\, c^4$**

Soln.: $(1 + a)\,(1 + b)\,(1+ c) - 1 = a + b + c + ab + bc + ca + abc$

$\geq 7\,(a \cdot b \cdot c \cdot ab \cdot bc \cdot ca \cdot abc)^{1/7}$ since A.M. $\geq$ G.M.

$= 7\,(a^4b^4c^4)^{1/7}$

$\therefore\ (1 + a)\,(1 + b)\,(1 + c) \geq 1 + 7\,(a^4b^4c^4)^{1/7} > 7\,(a^4b^4c^4)^{1/7}$

$\therefore\ [(1 + a)\,(1 + b)\,(1 + c)]^7 > 7^7a^4b^4c^4$.

20 **Find the sum of the products of integers 1, 2, 3, ..., n taken two at a time, and show that it is equal to half the excess of the sum of the cubes of the given integers over the sum of their squares.**

Soln.: For any numbers $a_1, a_2, \ldots, a_n$,

$$(a_1 + a_2 + \ldots.+ a_n)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{1\leq i<j\leq n} a_i a_j$$

$$\text{Hence } (1 + 2 + \ldots + n)^2 = \sum_{i-1}^{n} i^2 + 2\sum_{1\leq i<j\leq n} ij\,.$$

$$\text{The required sum} = \frac{1}{2}[(1+2+\ldots+n)^2 - (1^2+2^2+\ldots+n^2)]$$

$$= \frac{1}{2}[(1^3+2^3+\ldots+n^3)-(1^2+2^2+\ldots+n^2)]\,.$$

21) **Find the sum of** $\sum_{n=1}^{16} \frac{n^4}{4n^2-1}$

Soln.: $t_n = \frac{n^4}{4n^2-1} = \frac{16n^4-1+1}{16(4n^2-1)} = \frac{1}{16}(4n^2+1) + \frac{1}{16(4n^2-1)} = \frac{1}{16}(4n^2+1) + \frac{1}{32}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$

$$\sum_{n=1}^{16} t_n = \frac{1}{16}\left[\frac{4\cdot16\cdot17\cdot33}{6} + 16\right] + \frac{1}{32}\left(1 - \frac{1}{33}\right) = 375 + \frac{1}{33}.$$

22) **Find the sum to n terms of the series** $1+5\left(\frac{4n+1}{4n-3}\right)+9\left(\frac{4n+1}{4n-3}\right)^2+13\left(\frac{4n+1}{4n-3}\right)^3+\ldots$ **?**

Soln.: Let $x = \frac{4n+1}{4n-3}$, then $1 - x = \frac{-4}{4n-3} \Rightarrow \frac{1}{1-x} = -\frac{(4n-3)}{4}$

$$\frac{x}{1-x} = -\frac{(4n+1)}{4}$$

$S = 1 + 5x + 9x^2 + \ldots\ldots + (4n-3)x^{n-1}$

$Sx = x + 5x^2 + \ldots\ldots + (4n-3)x^n$

$S - Sx = 1 + 4x + 4x^2 + \ldots\ldots + 4x^{n-1} - (4n-3)x^n.$

$$S(1-x) = 1 + \frac{4x}{1-x}[1-x^{n-1}] - (4n-3)x^n$$

$$S = \frac{1}{1-x}\left[1 + \frac{4x}{1-x} - \frac{4x^n}{1-x} - (4n-3)x^n\right]$$

$$= -\frac{(4n-3)}{4}[1-(4n+1)+(4n-3)x^n-(4n-3)x^n] = n(4n-3).$$

23) **Calculate the value of** $\dfrac{\sum_{r=1}^{n} r^4}{\sum_{r=1}^{n} r^2}$

Soln.: $(x+1)^5 - x^5 = 5x^4 + 10x^3 + 10x^2 + 5x + 1$

$\Sigma[(k+1)^5 - k^5] = 5\,\Sigma k^4 + 10\,\Sigma k^3 + 10\,\Sigma k^2 + 5\,\Sigma k + \Sigma 1$

where Σ stands for $\sum_{k=1}^{n}$.

$$\therefore \quad (n+1)^5 - 1 = 5\sum k^4 + \frac{10n^2(n+1)^2}{4} + \frac{10n(n+1)(2n+1)}{6} + \frac{5n(n+1)}{2} + n$$

$$5\Sigma n^4 = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n - \frac{5}{2}n^2(n+1)^2 - \frac{5}{3}n(n+1)(2n+1) - \frac{5n(n+1)}{2} - n$$

$$= \frac{n(n+1)}{6}(6n^3+9n^2+n-1) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{6}$$

$$\therefore \quad \frac{\sum k^4}{\sum k^2} = \frac{3n(n+1)-1}{5}$$

Xtra SUPPLEMENT for competitive exams

Arithmetical Progression (A.P.)

- T_n of A.P. $= a+(n-1)d$
- S_n of A.P. $= \frac{n}{2}(a+l)$ or $\frac{n}{2}[2a+(n-1)d]$
- Common difference of A.P. is $d = T_n - T_{n-1}$ of all n.
- (a) $T_n = S_n - S_{n-1}$

  (b) $T_n = \frac{1}{2}\left[T_{n-k} + T_{n+k}\right], k < n$
- Arithmetic Mean (A.M.) between a and b is $A = \frac{a+b}{2}$
- Sum of n A.M.'s between a and $b = nA$. *i.e.* $A_1 + A_2 + ...A_n = nA$
- Any three numbers in A.P. can be taken as $a-d, a, a+d$
- Any four numbers in A.P. can be taken as $a-3d, a-d, a+d, a+3d$
- Five nos. in A.P. are $a-2d, a-d, a, a+d, a+2d$
- If p^{th} term of an A.P. is q and q^{th} term $= p$, then $T_{p+q} = 0, T_r = p+q-r$
- If $pT_p = qT_q$ of an $A.P.$, then $T_{p+q} = 0$
- If $S_p = q$ for an A.P., $S_q = p$, then $S_{p+q} = -(p+q)$
- If $S_p = S_q$ for an A.P., then $S_{p+q} = 0$
- $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the A.M. between a and b if $n = 0$

Geometrical Progression (G.P.)

- (a) $T_n = ar^{n-1}$

  (b) If l denotes the last term of a finite G.P., then n^{th} term from the end is $= \frac{l}{r^{n-1}}$
- $S_n = \frac{a(1-r^n)}{1-r}$, if $r < 1$ numerically $= \frac{a(r^n-1)}{r-1}$, if $r > 1$ numerically
- $S_\infty = \frac{a}{1-r}$ where $|r| < 1$.

  If $|r| \geq 1$, then S_∞ does not exist.
- Three numbers in G.P. can be taken as $\frac{a}{r}, a, ar$

  Five numbers in G.P. can be taken as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

Four numbers in G.P. can be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

- (i) If for a G.P., $T_p = P; T_q = Q$, then $T_n = \left[\frac{P^{n-q}}{Q^{n-p}}\right]^{\frac{1}{p-q}}$

  (ii) If for a G.P., $T_{m+n} = p; T_{m-n} = q$, then $T_m = \sqrt{pq}$; $T_n = p\left(\frac{q}{p}\right)^{\frac{m}{2n}}$

  (iii) If a, b, c are the $p^{\text{th}}, q^{\text{th}}, r^{\text{th}}$ terms of a G.P., then $a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = 1$

- (i) $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is G.M. between a and b if $n = -\frac{1}{2}$

  (ii) If A and G be the A.M. and G.M. between two numbers a, b are given by $A \pm \sqrt{(A+G)(A-G)}$

  [$\because\ x^2 - (a+b)x + ab = 0$]

  $\Rightarrow\ x^2 - 2Ax + G^2 = 0 \ \Rightarrow\ x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2} \ \Rightarrow\ x = A \pm \sqrt{(A+G)(A-G)}$]

- (i) $\sum n = \frac{n(n+1)}{2}$ (ii) $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$

  (iii) $\sum n^3 = \left(\sum n\right)^2 = \left[\frac{n(n+1)}{2}\right]^2$

  (iv) $S_n = \sum T_n = \sum\left(an^3 + bn^2 + cn + d\right) = a\sum n^3 + b\sum n^2 + c\sum n + dn$

- A series of the form $a + (a + d)\, r + (a + 2\, d)\, r^2 + (a + 3\, d)\, r^3 + \ldots + [a + (n - 1)\, d]\, r^{n-1} + \ldots$ is called **Arithmetico-Geometric series** (A.G. series)

- Summation of series as by the method of Differences,

  Let $S_n = T_1 + T_2 + T_3 + \ldots + T_n$ ...(1)

  Also $S_n = T_1 + T_2 + \ldots + T_{n-1} + T_n$ ...(2)

  (1)–(2) gives $0 = T_1 + (T_2 - T_1) + (T_3 - T_2) + \ldots + (T_n - T_{n-1}) + T_n$

Miscellaneous Xamples

1. If the n^{th} term of an A.P. is p, show that the sum of first $(2n–1)$ terms of the A.P. is $(2n – 1)\,p$.

Soln.: Let, a be the first term and d, the common difference of the A.P.

$\therefore$ n^{th} term $= a + (n–1)\, d = p$ ...(i)

$\therefore$ Sum of first $(2n – 1)$ terms $= \dfrac{(2n-1)}{2}\left[2a+(2n-1-1)d\right] = \dfrac{2n-1}{2}\left[2a+(2n-2)d\right]$

$= \dfrac{2n-1}{2}\times 2\left[a+(n-1)d\right]=(2n-1)p$ [from (i)]

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2. The 12^{th} term of a series in A.P. is –13 and the sum of first four terms of it is 24. Find the sum of its first 10 terms.

Soln.: Let a = first term of A.P.; d = common difference of A.P.

$\therefore$ 12^{th} term $= a + (12 – 1)\, d = a + 11\, d$.

$\therefore$ $a + 11d = –13$ ...(i)

Sum of first 4 terms $S_4=\dfrac{4}{2}[2a+(4-1)d]$

$\therefore$ $2\,(2a + 3d) = 24$ ...(ii)

Solving (i) & (ii), $a = 9$, $d = –2$.

$\therefore$ Sum of first ten terms, $S_{10}=\dfrac{10}{2}[2\times 9+(10-1)(-2)]$ $= 5\,(18 – 18) = 0$

---

3. The sum of n terms of two A.P. are in the ratio $\dfrac{7n+1}{4n+27}$. Find the ratio of their 11^{th} terms ?

Soln.: Let the two A.P. series have first terms a and b and common difference be c and d respectively.

$\therefore$ Sum of n terms of the two series are $S_1=\dfrac{n}{2}[2a+(n-1)c]$ and $S_2=\dfrac{n}{2}[2b+(n-1)d]$

$\therefore$ $\dfrac{S_1}{S_2}=\dfrac{2a+(n-1)c}{2b+(n-1)d}=\dfrac{7n+1}{4n+27}$ ...(i)

Again ratio of their 11^{th} terms is $\dfrac{a+(11-1)c}{b+(11-1)d}=\dfrac{a+10c}{b+10d}$ ...(ii)

By putting $n = 21$ in equation (i)

$\dfrac{2a+(21-1)c}{2b+(21-1)d}=\dfrac{2(a+10c)}{2(b+10d)}=\dfrac{7\times 21+1}{4\times 21+27}=\dfrac{4}{3}$

$\therefore$ Ratio of 11^{th} terms is 4 : 3

---

4. Find the sum of all natural numbers lying between 200 and 500, which are multiples of 3 or 7 or both.

Soln.: The first and last numbers between 200 and 500, which are divisible by 3 are 201 and 498.

$\therefore$ If there are n such numbers $498 = 201 + (n – 1) \times 3$ or $n = 100$.

$\therefore$ Sum $S_3=\dfrac{100}{2}[2\times 201+(100-1)\times 3]=50\times 699.$

Again, first and last numbers between 200 and 500, which are divisible by 7 are, 203 and 497.

$\therefore$ $497 = 203 + (n – 1) \times 7$ or $n = 43$

$\therefore$ $S_7=\dfrac{43}{2}[2\times 203+(43-1)\times 7]=43\times 350.$

Numbers which are divisible by 3 or 7 are all divisible by 3 × 7 *i.e.* 21.

First and last numbers which are divisible by 21 and lying between 200 and 500 are 210 and 483.

$\therefore\ 483 = 210 + (n - 1) \times 21$ or $n = 14$

$\therefore\ S_{21} = \frac{14}{2}[2\times 210+(14-1)\times 21] = 4851.$

Now, all numbers which are divisible by 21 are also divisible by both 3 and 7.

Hence, required sum $= S_3 + S_7 - S_{21} = 45159$

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5. Sum of some consecutive odd positive integers is $57^2 - 13^2$. Find the integers ?

Soln.: Let the number of odd integers be n and let the odd integers be

$(2m + 1), (2m + 3), (2m + 5), \ldots$

$\therefore$ Sum $= \frac{n}{2}[2(2m+1)+(n-1)\times 2] = n\,[2m + 1 + n - 1]$

$= n\,(2m + n) = 2mn + n^2 = (m^2 + 2mn + n^2) - m^2 = (m + n)^2 - m^2$

$\therefore\ 57^2 - 13^2 = (m + n)^2 - m^2 \quad \therefore\ m + n = 57$ and $m = 13 \quad \therefore\ n = 44$

Hence, the integers are, 27, 29, 31, ..., 113.

---

6. The sum of three numbers in G.P. is 35 and their product is 1000. Find the numbers.

Soln.: Let the three numbers in G.P. be $\frac{a}{r}, a, ar$

$\therefore\ \frac{a}{r}+a+ar=35$ ...(i)

$\frac{a}{r}\times a\times ar=1000$ ...(ii)

From (ii), $a^3 = 1000$. or $a = 10$

From (i), $\frac{10}{r} + 10 + 10r = 35$ or, $2r^2 - 5r + 2 = 0$

or, $(r - 2)(2r - 1) = 0 \quad \therefore\ r = 2$ or $\frac{1}{2}$

$\therefore$ The terms are $\frac{10}{2}$, 10, 10 × 2 or 5, 10, 20 or $\frac{10}{1/2}, 10, 10\times\frac{1}{2}$ *i.e.* 20, 10, 5.

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7. Calculate the sum to n terms : 4 + 44 + 444 + ...

Soln.: 4 + 44 + 444 + ... to n terms

$= 4\,[1 + 11 + 111 + \ldots$ to n terms$] = \frac{4}{9}\,[9 + 99 + 999 + \ldots$ to n terms$]$

$= \frac{4}{9}\,[(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots$ to n terms$]$

$= \frac{4}{9}\,[(10 + 10^2 + 10^3 + \ldots$ to n terms$) - (1 + 1 + 1 + \ldots$ to n terms$)]$

$$= \frac{4}{9}\left[\frac{10\cdot(10^n-1)}{10-1}-n\right] = \frac{4}{9}\left[\frac{10}{9}\left(10^n-1\right)-n\right]$$

---

8. Solve for x : $1 + a + a^2 + \cdots + a^{x-1} + a^x = (1 + a)(1 + a^2)(1 + a^4)(1 + a^8)$

Soln.: The series in L.H.S. is a G.P. with first term = 1 and C.R. = a, having $(x + 1)$ terms.

$\therefore\ \frac{(1-a^{x+1})}{1-a} = (1+a)(1+a^2)(1+a^4)(1+a^8)$

or, $1 - a^{x+1} = (1 - a^2)(1 + a^2)(1 + a^4)(1 + a^8) = 1 - a^{16}$

or, $a^{x+1} = a^{16}$ or, $x = 15$

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9. If $S_1, S_2, S_3, ..., S_n$ are the sums of n terms of n different G.P., whose first terms are 1 each and common ratios are 1, 2, 3, ..., n, show that $S_1 + S_2 + 2S_3 + 3S_4 + ... + (n-1)S_n = 1^n + 2^n + ... + n^n$.

Soln.: $S_1 = 1 + 1.1 + 1.1^2 + ... + 1.1^{n-1} = 1 \times n = n\ (a = 1, r = 1)$

$$S_2 = \frac{1(2^n - 1)}{2-1} = 2^n - 1$$

$$S_3 = \frac{1(3^n - 1)}{3-1} \quad \text{or,} \quad 2S_3 = 3^n - 1$$

$$S_4 = \frac{1(4^n - 1)}{4-1} \quad \text{or,} \quad 3S_4 = 4^n - 1$$

$$S_n = \frac{1(n^n - 1)}{n-1} \quad \text{or,} \quad (n-1)S_n = n^n - 1$$

Adding, $S_1 + S_2 + 2S_3 + 3S_4 + ...+ (n-1)S_n$

$$= n + (2^n - 1) + (3^n - 1) + (4^n - 1) + ... + (n^n - 1) = n + (2^n + 3^n + 4^n + ... + n^n) - 1(n-1) = 1^n + 2^n + ... + n^n.$$

---

10. Show that sum of first n terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + ...$ is $\frac{1}{2}n(n+1)^2$. If n is even integer.

Soln.: Since n is an even integer, let $n = 2p$.

$\therefore$ Given series $= (1^2 + 3^2 + 5^2 + ...$ to p terms$) + 2\,(2^2 + 4^2 + ...$ to p terms$)$

$= S_1 + 2S_2$ (say)

$\therefore$ For S_1, $t_n = [1 + (n-1)\,2]^2 = (2n-1)^2 = 4n^2 - 4n + 1$.

$\therefore$ $t_1 = 4.1^2 - 4.1 + 1;$

$t_2 = 4.2^2 - 4.2 + 1;$

....

$t_p = 4.p^2 - 4.p + 1;$

Adding, $S_1 = 4\,(1^2 + 2^2 + ...$ to p terms$) - 4\,(1 + 2 + ...$ to p terms$) + p$.

$$= 4 \cdot \frac{p(p+1)(2p+1)}{6} - 4 \cdot \frac{p(p+1)}{2} + p = p\,(4p^2 + 3p - 4)$$

For S_2, $t_n = (2n)^2 = 4n^2$ $\therefore$ $S_2 = 4\,(1^2 + 2^2 + ...$ to p terms$)$

Adding get the answer.

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11. If n^{th} term of a series is given by $\frac{n^4 + 2n^3 - n - 1}{n(n+1)}$, find the sum of its first n terms.

Soln.: Given, $t_n = \frac{n^4 + 2n^3 - n - 1}{n(n+1)} = \frac{(n^4 + n^3) + (n^3 - n) - 1}{n\,(n+1)}$

$$= \frac{n^3(n+1) + n(n^2 - 1) - 1}{n(n+1)} = (n^2 + n - 1) - \frac{1}{n(n+1)} = n^2 + n - 1 - \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\therefore \quad S = (1^2 + 2^2 + ... + n^2) + (1 + 2 + 3 + ... + n) - n - \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right]$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} - n - \left[1 - \frac{1}{n+1}\right]$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} - n - \frac{n}{(n+1)} = \frac{n(n+2)(n^2 + 2n - 2)}{3(n+1)}$$

12. Find the sum of the numbers in the n^{th} bracket of the series (1) + (2 + 3 + 4) + (5 + 6 + 7 + 8 + 9) + ...

Soln.: First term in first group = 1, no. of terms = 1.

First term in second group = 2, no. of terms = 3.

First term in third group = 5, no. of terms = 5.

$\therefore$ Number of terms in n^{th} group = n^{th} term of A.P. 1, 3, 5, ... = $1 + (n-1) \times 2 = 2n - 1$.

Let t_n be the first term of n^{th} group.

$\therefore t_n = n^{th}$ term of the series 1, 2, 5, 10, ...

Let $S = 1 + 2 + 5 + 10 + ... + t_n$. ....(i)

$S = 1 + 2 + 5 + ... + t_{n-1} + t_n$. ....(ii)

Subtracting (ii) from (i) $0 = 1 + (1 + 3 + 5 + ...$ to $(n-1)$ terms$) - t_n$.

or, $t_n = 1 + \frac{n-1}{2}[2\times1+(n-1-1)\times2] = 1+(n-1)^2$

C.D. in the n^{th} group = 1

$\therefore$ Sum of term in n^{th} group $= \frac{2n-1}{2}\left[2\left\{1+(n-1)^2\right\}+(2n-1-1)\times1\right] = (2n-1)(n^2-n+1)$

---

13. In an A.P. consisting of 20 terms, sum of the terms in even places is 250 and that of the terms in the odd places is 220. Find the two middle terms of the progression.

Soln.: If first term = a and C.D. = d,

The series is $a, (a + d), (a + 2d), (a + 3d), (a + 4d), ..., (a + 19d)$

$\therefore$ $(a + d) + (a + 3d) + (a + 5d) + ... + (a + 19d) = 250$ ...(i)

and $a + (a + 2d) + (a + 4d) + ... + (a + 18d) = 220$ ...(ii)

From (i), $10a + [d + 3d + 5d + ... + 19d] = 250$

The series in the bracket is an A.P. with first term = d and common difference = $2d$, having 10 terms.

$\therefore$ $10a + \frac{10}{2}[2d + (10-1)\times 2d] = 250$

or, $10a + 100d = 250$ ...(iii)

From (ii), $10a + [2d + 4d + ... + 18d] = 220$

The series in the bracket is an A.P. with first term = $2d$, common difference = $2d$, having 9 terms.

$\therefore$ $10a + \frac{9}{2}[2\times 2d + (9-1)2d] = 220$ or, $10a + 90d = 220$ ...(iv)

From (iii) and (iv), $d = 3$, $a = -5$.

Middle term of the 20 term series are 10^{th} & 11^{th} terms.

$\therefore$ $t_{10} = -5 + (10 - 1) \times 3 = 22$

$t_{11} = -5 + (11 - 1) \times 3 = 25$

---

14. Find sum to n terms : $\frac{1}{1\cdot2\cdot3\cdot4} + \frac{1}{2\cdot3\cdot4\cdot5} + \frac{1}{3\cdot4\cdot5\cdot6} + ...$

Soln.: The r^{th} term and $(r + 1)^{th}$ term of the series can be written as

$t_r = \frac{1}{r(r+1)(r+2)(r+3)}$ and $t_{r+1} = \frac{1}{(r+1)(r+2)(r+3)(r+4)}$

Dividing, we get, $rt_r = (r + 4).\ t_{r+1}$ or, $r \cdot t_r - (r + 1)\ t_{r+1} = 3\ t_{r+1}$

Putting $r = 1, 2, 3,$ .... to $(n - 1)$, we get,

$t_1 - 2t_2 = 3t_2$

$2t_2 - 3t_3 = 3t_3$

$3t_3 - 4t_4 = 3t_4$

....

$(n-1)\ t_{n-1} - n \cdot t_n = 3t_n$.

Adding $t_1 - n \cdot t_n = 3\,[\,t_2 + t_3 + \ldots\ldots.. + t_n]$
or, $4t_1 - n \cdot t_n = 3S$. (Adding $3t_1$ to both sides of the eqn.)

$$\therefore \quad S = \frac{4}{3}t_1 - \frac{n}{3}\cdot t_n = \frac{4}{3}\times\frac{1}{1\cdot 2\cdot 3\cdot 4} - \frac{n}{3n(n+1)(n+2)(n+3)} = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}.$$

---

15. If $(1^2 - t_1) + (2^2 - t_2) + \cdots\cdots + (n^2 - t_n) = \frac{1}{3}n(n^2 - 1)$, prove that $t_n = n$.

Soln.: L.H.S. $= (1^2 + 2^2 + 3^2 + \ldots\ldots. + n^2) - (t_1 + t_2 + t_3 + \ldots\ldots.. + t_n) = \dfrac{n(n+1)(2n+1)}{6} - S_n$

$$\Rightarrow \quad \frac{n(n+1)(2n+1)}{6} - S_n = \frac{1}{3}n(n^2 - 1)$$

$$\therefore \quad S_n = \frac{n(n+1)(2n+1)}{6} - \frac{1}{3}\cdot n(n+1)(n-1) = \frac{n(n+1)}{3}\left[\frac{2n+1}{2} - (n-1)\right] = \frac{n(n+1)}{2}$$

Sum of n natural numbers $= 1 + 2 + 3 + \ldots\ldots + n$.
$\therefore \quad n^{\text{th}}$ term $t_n = n$.

---

16. In the following two A.P.'s how many terms are identical?
2, 5, 8, 11, ... to 60 terms and 3, 5, 7, .. to 50 terms.

Soln.: If a, $a + d$, ... be an A.P. then its n^{th} term is given by $T_n = a + (n - 1)d$
If possible, let the p^{th} term of the 1^{st} A.P. be identical with q^{th} term of the 2^{nd} A.P.
Then $2 + (p - 1)\,3 = 3 + (q - 1)\,2$
$\Rightarrow\ 3p - 1 = 2q + 1 \ \Rightarrow\ 3p = 2(q + 1)$ ...(i)
$1 \le p \le 60;\ 1 \le q \le 50,\ p, q \in N$
From (i), we conclude that p is a multiple of 2
$\therefore$ let $p = 2s$
$\because\ 1 \le p \le 60 \quad \therefore\ 1 \le s \le 30$
Putting $p = 2s$ in (i) $3s = q + 1$
As $1 \le q \le 50 \ \therefore\ 1 \le s \le 17$
So, there are 17 terms identical.

---

17. Does there exist a geometric progression containing 27, 8 and 12 as three of its terms? If it exists, how many such progressions are possible?

Soln.: The n^{th} term of a G.P. a, ar, ar^2 ... is $T_n = ar^{n-1}$.
Let the terms be in G.P. of which 8 is the first term and $T_m = 12$, $T_n = 27$.
Let r be the common ratio, $r \ne \pm 1$
$12 = 8r^{m-1}$, $27 = 8r^{n-1}$

$$\Rightarrow \quad \left(\frac{3}{2}\right) = r^{m-1},\ \left(\frac{3}{2}\right)^3 = r^{n-1} \quad \Rightarrow \quad (r)^{3(m-1)} = r^{n-1}$$

On equating the powers of r, we get

$$\Rightarrow\ 3m - 3 = n - 1 \ \Rightarrow\ 3m = n + 2 \Rightarrow \frac{m}{1} = \frac{n+2}{3} = k \text{ (say)} \ \Rightarrow\ m = k,\ n = 3k - 2$$

For $k = 2, 3, 4, \ldots$ we get sets of distinct positive integral values of m and n, each of which gives a G.P. of which 8, 12, 27 are three terms.

---

18. Prove that the numbers 49, 4489, 444889, ... obtained by inserting 48 into the middle of the preceding number are square of integers.

Soln.: $49 = 40 + 8 + 1 = 4 \times 10 + 8 + 1$
$4489 = 4000 + 400 + 80 + 8 + 1 = 4(10^3 + 10^2) + 8(10 + 1) + 1$
$444889 = 4(10^5 + 10^4 + 10^3) + 8(10^2 + 10 + 1) + 1$
$= 4 \times 10^3(1 + 10 + 10^2) + 8(1 + 10 + 10^2) + 1$

$$= (4 \times 10^3 + 8)\,\frac{(10^3 - 1)}{10 - 1} + 1 = \frac{4 \times 10^6 + 4 \times 10^3 + 1}{9} = \left(\frac{2 \times 10^3 + 1}{3}\right)^2$$

In the general case,

$$\underbrace{444...4}_{n \text{ digits}}\underbrace{888...89}_{n \text{ digits}} = \underbrace{44...4}_{n \text{ digits}}\underbrace{88...8 + 1}_{n \text{ digits}}$$

$= 4 \times 10^n(1 + 10 + 10^2 + ... + 10^{n-1}) + 8(1 + 10 + ... + 10^{n-1}) + 1$

$$= (4 \times 10^n + 8)\left(\frac{10^n - 1}{10 - 1}\right) + 1 = \frac{4 \times 10^{2n} + 4 \times 10^n + 1}{9} = \left(\frac{2 \times 10^n + 1}{3}\right)^2$$

and $\left(\frac{2 \times 10^n + 1}{3}\right)$ is always an integer.

---

19. Let the angles of a triangle ABC be in A.P. and let $b : c = \sqrt{3} : \sqrt{2}$. Find the angle A.

Soln.: $A + B + C = 180°$, $2B = A + C$ ($\because A, B, C$ are in A.P.)

$\therefore\ 3B = 180°$ or $B = 60°$

Given $\frac{b}{\sqrt{3}} = \frac{c}{\sqrt{2}} = k \quad \therefore \quad b = k\sqrt{3},\ c = k\sqrt{2}$

By the sine rule, $\frac{b}{\sin B} = \frac{c}{\sin C}$; $\frac{k\sqrt{3}}{\sin 60°} = \frac{k\sqrt{2}}{\sin C}$

$\therefore \quad \sin C = \frac{1}{\sqrt{2}}$

$C = 45°$ and $C = 135°$ is impossible

$A = 180° - (B + C) = 180° - (60° + 45°) = 75°$.

---

20. Find three numbers a, b, c between 2 and 18 such that (i) their sum is 25 (ii) 2, a, b are consecutive terms of an A.P. (iii) the numbers b, c, 18 are consecutive terms of a G.P.

Soln.: $a + b + c = 25$ .... (1)

$2a = b + 2$ .... (2)

$c^2 = 18b$ .... (3)

$\Rightarrow \frac{b+2}{2} + b + \sqrt{18b} = 25 \quad \Rightarrow 3b + 6\sqrt{2b} - 48 = 0$

$\Rightarrow b + 2\sqrt{2b} - 16 = 0 \quad \Rightarrow b + 4\sqrt{2}\sqrt{b} - 2\sqrt{2}\sqrt{b} - 16 = 0$

$\Rightarrow (\sqrt{b} - 2\sqrt{2})(\sqrt{b} + 4\sqrt{2}) = 0 \quad \Rightarrow b = 8, a = 5, c = 12$

---

21. If $\log_3 2$, $\log_3(2^x - 5)$ and $\log_3\left(2^x - \frac{7}{2}\right)$ are in arithmetic progression, determine the value of x.

Soln.: $2\log_3(2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right) \quad \Rightarrow \quad (2^x - 5)^2 = 2\left(2^x - \frac{7}{2}\right)$

$\therefore \quad (2^x)^2 - 10 \times 2^x + 25 = 2 \times 2^x - 7$ is a quadratic equation in 2^x.

$\therefore \quad (2^x)^2 - 12\ 2^x + 32 = 0 \Rightarrow (2^x - 4)(2^x - 8) = 0 \Rightarrow 2^x = 4$ or $2^x = 8 \Rightarrow x = 2$ or 3

For $x = 2$, $\log_3(2^x - 5)$ is not meaningful.

$\therefore \quad x = 3$ is the real solution.

---

22. The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is square of an integer.

Soln.: Let four consecutive terms of the A.P. be $a - 3d$, $a - d$, $a + d$ and $a + 3d$. Common difference is $2d$. Given $a - 3d$, $a - d$, $a + d$ and $a + 3d$ are integers. Therefore, $2d$ is also an integer.

Now, $E = (a - 3d)(a - d)(a + d)(a + 3d) + (2d)^4 = (a^2 - 9d^2)(a^2 - d^2) + 16d^4$

$= a^4 - 10d^2a^2 + 9d^4 + 16d^4$

$E = (a^2 - 5d^2)^2$ is an integer (As $a - 3d$, $a + 3d$ and $2d$ are integers $\Rightarrow a^2 - 5d^2$ is also an integer)

Thus, E is the square of an integer.

---

23. If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + ... + (-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$, then find the least natural number n_0 such that $b_n > a_n \forall n \geq n_0$.

Soln.: $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + ...(-1)^{n-1}\left(\frac{3}{4}\right)^n = \frac{\frac{3}{4}\left(1 - \left(-\frac{3}{4}\right)^n\right)}{1 + \frac{3}{4}} = \frac{3}{7}\left(1 - \left(-\frac{3}{4}\right)^n\right)$

$b_n = 1 - a_n$ and $b_n > a_n \ \forall\, n \geq n_0$

$\Rightarrow \frac{6}{7}\left[1 - \left(-\frac{3}{4}\right)^n\right] < 1$

$\Rightarrow -\left(-\frac{3}{4}\right)^n < \frac{1}{6} \Rightarrow (-3)^{n+1} < 2^{2n-1}$

for n to be even, inequality always holds. For n to be odd, it holds for $n \geq 6$

$\therefore$ The least natural number for which it holds is 6

($\because$ it holds for every even natural number.)

---

24. Find the sum to n terms: $2 + 5 + 14 + 41 + ...$

Soln.: Since the series is neither an A.P. nor a G.P. thus we will proceed towards its difference series.

Determination of t_n (n^{th} term)

$S_n = 2 + 5 + 14 + 41 + ... + T_{n-1} + T_n$ ....(i)

Also $S_n = \quad 2 + 5 + 14 + ... + T_{n-2} + T_{n-1} + T_n$ ....(ii)

On subtracting (ii) from (i) we get $0 = 2 + [3 + 9 + ... + (n - 1) \text{ term}] - T_n$

$\Rightarrow T_n = 2 + [\text{sum of G.P. with } a = 3, r = 3] = 2 + \frac{3(3^{n-1} - 1)}{3 - 1} = 2 + \frac{3(3^{n-1} - 1)}{2}$

$= \frac{1}{2}[4 + 3^n - 3] = \frac{1}{2}[3^n + 1]$

$\therefore$ n^{th} term $T_n = \frac{1}{2}[1 + 3^n]$

Determination of $S_n = \Sigma T_n$

$S_n = \Sigma T_n = \frac{1}{2}\Sigma 1 + \frac{1}{2}\Sigma 3^n = \frac{n}{2} + \frac{1}{2} 3.\frac{(3^n - 1)}{3 - 2} = \frac{n}{2} + \frac{3}{4}(3^n - 1)$

$\therefore$ $S_n = \frac{n}{2} + \frac{3}{4}(3^n - 1)$.

---

25. Find the sum of the series: $2 + 4 + 7 + 11 + 16 + ...$ upto n terms.

Soln.: Since the terms of the given series are neither in A.P. nor in G.P., so we will proceed as in the case of difference series.

Determination of n^{th} term T_n

$S = 2 + 4 + 7 + 11 + 16 + ... T_{n-1} + T_n$

$S = \quad 2 + 4 + 7 + 11 + ... T_{n-2} + T_{n-1} + T_n$

On subtraction, we get $0 = 2 + [2 + 3 + 4 + 5 + ... + T_n - T_{n-1}] - T_n$

$T_n = 1 + [1 + 2 + ... n \text{ term}] = 1 + \frac{1}{2}(n + 1)(n) = 1 + \frac{1}{2}n^2 + \frac{n}{2}$

$\therefore$ $T_n = 1 + \frac{1}{2}n^2 + \frac{n}{2}$

Summation of Series

$S_n = \Sigma T_n = \Sigma 1 + \Sigma \frac{1}{2} n^2 + \Sigma \frac{n}{2} = n + \frac{1}{2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}.\frac{n(n+1)}{2} = \frac{n(n^2 + 3n + 8)}{6}$

26. If $S_1, S_2, S_3, ...$ are the sums of n terms of A.P.'s whose first terms are 1, 2, 3... and common differences are 1, 3, 5, ... respectively, show that $S_1 + S_2 + ... S_m = \frac{1}{2} mn\ (mn + 1)$.

Soln.: $S_n = \frac{n}{2}[2a + (n - 1)\ d]$.

$S_r = \frac{n}{2}[2r + (n - 1)\ (2r - 1)] \quad \Rightarrow S_r = n^2 r - \frac{n}{2}(n - 1)$

$\therefore\ S_1 + S_2 + ... + S_m = \sum_{r=1}^{m} S_r = \sum_{r=1}^{m} n^2 r - \sum_{r=1}^{m} \frac{n}{2}(n-1) = n^2 . \frac{m(m+1)}{2} - \frac{n(n-1)}{2}.m = \frac{mn}{2}[mn+1]$.

---

27. Show that the sum of the product of first n natural numbers, taken two at a time, is equal to $\frac{1}{24} n(n+1)(n-1)(3n+2)$.

Soln.: $S = (1 \cdot 2 + 1 \cdot 3 + ... + 1 \cdot n) + (2 \cdot 3 + 2 \cdot 4 + ... + ... (n-1) \cdot n.$ *i.e.* $S = \sum_{j=i}^{n} \sum_{\substack{i=1 \\ i<j}}^{n} (i \cdot j)$

We know that $(1 + 2 + ... + n)^2 = 1^2 + 2^2 + ... + n^2 + 2\sum_{j=1}^{n} \sum_{\substack{i=1 \\ i<j}}^{n} i.j$.

$\sum_{j=1}^{n} \sum_{\substack{i=1 \\ i<j}}^{n} i.j = \frac{1}{2}[(1 + 2 + ... + n)^2 - (1^2 + 2^2 + ... + n^2)]$

$S = \frac{1}{2}\left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right]$ or $S = \frac{1}{24} n(n + 1)\ [3n\ (n + 1) - 2\ (2n + 1)]$

$S = \frac{1}{24} n(n + 1)\ (n - 1)\ (3n + 2)$.

---

28. Sum to infinite terms: $1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + ...\infty\ (|\ x\ | < 1)$.

Soln.: This is a special case of difference series.

In such case for $S_n = p_0 + p_1 x + p_2 x^2 + ... + p_n x^n$

We multiply the series by x, and then subtract the two series. The step is applied repeatively on the resultant series till we get some constant series, A.P. a G.P. for which sum can be determined easily.

$S = 2 + 6x + 12x^2 + 20x^3 + ...xS = 2x + 6x^2 + 12x^3 + ...$

On subtraction we get; $(1 - x)S = 2 + 4x + 6x^2 + 8x^3 + ...$ ...(1)

Again multiplying by x we get $(1 - x)\ xS = 2x + 4x^2 + 6x^3 + ...$ ...(2)

On subtraction of (2) from (1) we get; $(1 - x)^2 S = 2 + 2x + 2x^2 + ... = 2 + 2(x + x^2 + ...)$

G.P. with $|x| < 1$

$\Rightarrow\ (1 - x)^2 S = 2 + \frac{2x}{1-x}$ (sum of G.P. of infinite term)

$\Rightarrow\ S = \frac{2}{(1-x)^3}$.

---

29. The sum of an infinite geometric series is 2 and the sum of the geometric series made from the cubes of this infinite series is 24. Then, find the series.

Soln.: Given a G.P. whose sum of infinite terms is 2. Also, the sum of the G.P. having the cubes of the terms of the original series is 24.

S_∞ (sum of infinite terms of G.P.) $= \frac{a}{1-r}$, Where a is first term and r is common ratio.

Let the original G.P. be $a, ar, ar^2, ...$ ..(1)

$\therefore$ The new cubic series is $a^3, a^3r^3, a^3r^6, ...$ ...(2)

S_∞ for (1) is $\frac{a}{1-r} = 2$ (given) and S_∞ for (2) is $\dfrac{a^3}{1-r^3}$ = 24 (given) $\therefore$ $\dfrac{1-r^3}{(1-r)^3} = \dfrac{1}{3}$

$\therefore$ $1 - 2r + r^2 = 3(1 + r + r^2)$ or $2r^2 + 5r + 2 = 0$ $\therefore$ $r = -2$ or $-1/2$

Since $|r| < 1$ $\Rightarrow$ $r = -1/2$

Putting this value we get $a = 3$, $\therefore$ Series is $3 - \dfrac{3}{2} + \dfrac{3}{4} - \dfrac{3}{8} \ldots$

---

30. The sum of three numbers in G.P. is 14. If the first two terms are each increased by 1 and the third term decreased by 1, the resulting numbers are in A.P. Find the numbers.

Soln.: Given that sum of three numbers in G.P. is 14. Also when the first two numbers increased by 1 and last term is decreased by 1, an A.P. is formed.

When required the three terms in G.P. has to be taken a, ar, ar^2.

Given that $a + ar + ar^2 = 14 \Rightarrow a(1 + r + r^2) = 14$ ...(1)

Also $a + 1$, $ar + 1$, $ar^2 - 1$ forms an A.P.

so $2(ar + 1) = a + 1 + ar^2 - 1 = a + ar^2$

$\therefore$ $2(ar + 1) + ar = a + ar + ar^2 = 14$ (from (1))

$\Rightarrow$ $3ar + 2 = 14 \Rightarrow ar = 4$ ...(2)

$\Rightarrow$ $a = 4/r$

$\therefore$ Substituting in (1) we get; $2r^2 - 5r + 2 = 0 \Rightarrow r = 2, 1/2$

$\therefore$ We get the G.P. as 2, 4, 8 or 8, 4, 2.

---

31. If a, b, c are in G.P., then show that the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\dfrac{d}{a}, \dfrac{e}{b}, \dfrac{f}{c}$ are in A.P.

Soln.: Since a, b, c are in G.P.

$\Rightarrow$ $b^2 = ac \Rightarrow b = \sqrt{ac}.$

Since a, b, c are in G.P.

Substituting in $ax^2 + 2bx + c = 0$ we get; $ax^2 + 2\sqrt{ac}x + c = 0$

$\Rightarrow$ $\left(\sqrt{a}x + \sqrt{c}\right)^2 = 0 \Rightarrow x = -\sqrt{c}/\sqrt{a}$ is only root.

Since two equations have a common root

Thus $x = -\dfrac{\sqrt{c}}{\sqrt{a}}$ must be the root of $dx^2 + 2ex + f = 0$

$$\Rightarrow \quad d\left(-\frac{\sqrt{c}}{\sqrt{a}}\right)^2 + 2e\left(\frac{-\sqrt{c}}{\sqrt{a}}\right) + f = 0 \Rightarrow \frac{dc}{a} - 2e\frac{\sqrt{c}}{\sqrt{a}} + f = 0$$

$$\Rightarrow \quad \frac{d}{a} - \frac{2e}{\sqrt{ac}} + \frac{f}{c} = 0 \Rightarrow \frac{d}{a} + \frac{f}{c} = \frac{2e}{b}$$

$\therefore$ $\dfrac{d}{a}, \dfrac{e}{b}, \dfrac{f}{c}$ are in A.P.

---

32. Show that if the lengths of sides of a right angles triangle are in A.P., then their ratio is 3 : 4 : 5.

Soln.: Given a right angle Δ whose sides are in A.P.

When required we take the three terms in A.P. as $a - d$, a, $a + d$.

Let $a - d$, a, $a + d$ are the three sides of the given right angled triangle.

$\Rightarrow$ $a + d$ must be the hypotenuse.

Now by Pythagoras theorem, $(a + d)^2 = a^2 + (a - d)^2$

$\Rightarrow$ $a^2 + d^2 + 2ad = a^2 + a^2 + d^2 - 2ad \Rightarrow a^2 = 4ad \Rightarrow d = a/4$

So three sides are $a - a/4 = 3a/4$, $a/4$ and $5a/4$

$\therefore$ Their ratio is 3 : 4 : 5.

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33. The sum of four numbers in A.P. is 48 and the ratio of the product of the extremes to the product of the means is 27 : 35. Find A.P.?

Soln.: Given an A.P. of four terms whose sum of terms is 48. Also ratio of product of first and last term to the product of means is 27 : 35.

We know that sum of n terms of A.P., $S_n = \frac{n}{2}(2a + (n - 1)d)$

Also, when required we take the four terms of an A.P. as $a - 3d$, $a - d$, $a + d$, $a + 3d$.

Given that sum is 48 $\Rightarrow$ $a - 3d + a - d + a + d + a + 3d = 4a = 48 \Rightarrow a = 12$

Also, $\frac{(a-3d)(a+3d)}{(a-d)(a+d)} = \frac{27}{35}$ ($\because$ Mean of first and third term is second term and that of second and last term is third term)

$\Rightarrow$ $\frac{a^2 - 9d^2}{a^2 - d^2} = \frac{27}{35}$ or $35a^2 - 35 \times 9\ d^2 = 27a^2 - 27d^2$

$\Rightarrow$ $8a^2 = (35 \times 9 - 27)\ d^2 = 288\ d^2$

$\therefore$ $d = \sqrt{\frac{8}{288}}a = \sqrt{4} = \pm 2$ $\quad\therefore$ $d = \pm 2$

$\therefore$ We get the progression as either 6, 10, 14, 18 or 18, 14, 10, 6.

EXERCISE

Multiple Choice Questions

1. a, b, c are three distinct real numbers, which are in G.P. and $a + b + c = xb$. Then

(a) $x < -1$ or $x > 3$ (b) $-1 < x < 3$

(c) $-1 < x < 2$ (d) $0 < x < 1$

2. Let $S = \frac{4}{19} + \frac{44}{19^2} + \frac{444}{19^3} + ...\infty$. Then S is equal to

(a) $\frac{38}{81}$ (b) $\frac{4}{19}$

(c) $\frac{36}{171}$ (d) none of these

3. If a be a positive real number and A.M of a and $2a$ exceeded the G.M. by 2, then a is equal to

(a) $3(4 \pm \sqrt{2})$ (b) $3 \pm 2\sqrt{2}$

(c) $4(3 \pm 2\sqrt{2})$ (d) $4 \pm 2\sqrt{2}$

4. If 1, $\log_{81}(3^x + 48)$ and $\log_9\left(3^x - \frac{8}{3}\right)$ are in A.P., then x is equal to

(a) 1 (b) 2

(c) 9 (d) 3

5. If $\log_{2^{1/2}} a + \log_{2^{1/4}} a + \log_{2^{1/6}} a + \log_{2^{1/8}} a + ...$ upto 20 terms is 840, then a is equal to

(a) 2 (b) 1

(c) 4 (d) $\sqrt{2}$

6. The number of divisors of 72, 2025 and 1568 are in

(a) A.P. (b) G.P

. (c) A.G.P. (d) none of these

7. The sum to infinity of the series

$1 + 2\left(1 - \frac{1}{n}\right) + 3\left(1 - \frac{1}{n}\right)^2 + ...$ where $n \in N$, is given by

(a) $n(n - 1)$ (b) $n\left(1 - \frac{1}{n}\right)^2$

(c) n^2 (d) $\left(\frac{n-1}{n}\right)^2$

8. If a_1, a_2, a_3 (with $a_1 > 0$) are in G.P. with common ratio r, then the value of r with which the inequality $9a_1 + 5a_3 > 14a_2$ holds, cannot lie in the interval

(a) $\left[1, \frac{9}{2}\right]$ (b) $(-\infty, 0)$

(c) $\left[\frac{5}{9}, 1\right]$ (d) $\left[1, \frac{9}{5}\right]$

9. If a_1, a_2, a_3, a_4 and b are real numbers such that $(a_1^2 + a_2^2 + a_3^2)b^2 - 2(a_1a_2 + a_2a_3 + a_3a_4)b + (a_2^2 + a_3^2 + a_4^2) \le 0$, then a_1, a_2, a_3, a_4 are

(a) in A.P.

(b) in G.P.

(c) in A.G.P.

(d) such that $(a_1 + a_2)(a_3 - a_4) = (a_1 + a_3)(a_2 - a_4)$

10. If a, b, c are positive numbers in A.P., such that their product is 64, then the minimum value of b equal to

(a) 2 (b) 4

(c) 1 (d) does not exist

11. If $a_1, a_2, a_3, ..., a_n$ are n distinct odd numbers not divisible by any prime greater than 5, then $\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}$

(a) < 1 (b) = 1

(c) = 2 (d) < 2

12. If $\sin\theta$, $\cos\theta$, $\tan\theta$ are in G.P., then $\cot^6\theta - \cot^2\theta =$

(a) 0 (b) 1

(c) 4 (d) 2

13. The sum to n terms of the series $\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + ...$ is equal to

(a) $\frac{3n}{n+1}$ (b) $\frac{6n}{n+1}$

(c) $\frac{9n}{n+1}$ (d) $\frac{12n}{n+1}$

14. $2^{1/4} \cdot 4^{1/8} \cdot 8^{1/16} \cdot 16^{1/32}$ ... is equal to

(a) 1 (b) 2

(c) 3/2 (d) 5/2

15. If $\sum_{j=1}^{21} a_j = 693$, where $a_1, a_2, ..., a_{21}$ are in A.P., then $\sum_{i=0}^{10} a_{2i+1}$ is equal to

(a) 361 (b) 396
(c) 363 (d) data is insufficient

16. If the ratio of the sums of m and n terms of an A.P. is $m^2 : n^2$, then the ratio of its m^{th} and n^{th} terms is given by

(a) $2m + 1 : 2n + 1$ (b) $2m - 1 : 2n - 1$
(c) $m : n$ (d) $m - 1 : n - 1$

17. The natural numbers are grouped as follows: $S_1 = \{1\}$, $S_2 = \{2, 3, 4\}$, $S_3 = \{5, 6, 7, 8, 9\}$, ..., then the first element of S_{21} is

(a) 391 (b) 399
(c) 401 (d) 442

18. Three numbers, the third of which being 12, form a decreasing G.P. If the third term were 9 instead of 12, the three numbers would have formed an A.P. Then the common ratio of the original G.P., is

(a) $\frac{3}{7}$ (b) $\frac{2}{3}$
(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

19. The sum of the series
$1 \cdot n + 2 \cdot (n - 1) + 3 \cdot (n - 2) + ... + n \cdot 1 =$

(a) $\frac{n(n+1)(2n+1)}{6}$ (b) $\frac{n(n+1)(n+2)}{6}$
(c) $\frac{n(n+1)(n+2)}{3}$ (d) $\frac{n(n+1)(2n+1)}{3}$

20. If $21(x^2 + y^2 + z^2) = (x + 2y + 4z)^2$, then x, y, z are in

(a) A.P. (b) G.P.
(c) A.G.P. (d) none of these

21. The sum of the series
$\frac{2}{1\cdot 2}+\frac{5}{2\cdot 3}\cdot 2+\frac{10}{3\cdot 4}\cdot 2^2+\frac{17}{4\cdot 5}2^3+.....$ to n terms is

(a) $\frac{n}{n+1}2^{n+1}$ (b) $\frac{n+1}{n}2^{n+1}$
(c) $\frac{n}{n+1}2^{n}$ (d) $\frac{n+1}{n}2^{n}$

22. The value of $(n - 2)^2 + (n - 4)^2 + (n - 6)^2 + ...$ to n terms is

(a) $\frac{n}{3}(n^2 + 2)$ (b) $\frac{n}{2}(n^2 + 3)$
(c) $\frac{n}{3}(n^2 - 2)$ (d) $\frac{n}{2}(n^2 - 3)$

23. Three non-zero real numbers form an A.P. and the square of the numbers taken in the same order constitute a G.P. Then the number of all possible common ratios of the G.P. are

(a) 1 (b) 2
(c) 3 (d) 4

24. If $x^{15} - x^{13} + x^{11} - x^9 + x^7 - x^5 + x^3 - x = 7$, then

(a) x^{16} is equal to 15
(b) x^{16} is less than 15
(c) x^{16} is greater than 15
(d) nothing can be said regarding the value of x^{16}

25. The 6^{th} term of an A.P. is 18 and the 9^{th} term is 12. The 15^{th} term is equal to

(a) 0 (b) 1
(c) 2 (d) 3

26. The arithmetic mean of two numbers is 3 times their geometric mean and the sum of the squares of the two numbers is 34. The two numbers are

(a) $2\sqrt{3}+\sqrt{5}, 2\sqrt{3}-\sqrt{5}$ (b) $3+2\sqrt{2}, 3-2\sqrt{2}$
(c) $\sqrt{10}+\sqrt{7}, \sqrt{10}-\sqrt{7}$ (d) none of these

27. Four geometric means are inserted between the numbers $2^{11} - 1$ and $2^{11} + 1$. The product of these geometric means is

(a) $2^{44} - 2^{23} + 1$ (b) $2^{44} - 2^{22} + 1$
(c) $2^{22} - 2^{11} + 1$ (d) $2^{22} - 2^{12} + 1$.

28. Let t_r denotes the r^{th} term of an A.P. Also suppose that $t_m = \frac{1}{n}$ and $t_n = \frac{1}{m}$, $(m \neq n)$, for some positive integers m and n, then which of the following is necessarily a root of the equation $(l + m - 2n)x^2 + (m + n - 2l)x + (n + l - 2m) = 0$

(a) t_n (b) t_m
(c) t_{m+n} (d) t_{mn}

29. The sum of the A.M. and G.M. of two positive numbers is equal to the difference between the numbers. The numbers are in the ratio

(a) 1 : 3 (b) 1 : 6
(c) 9 : 1 (d) 1 : 12

30. $(2n + 1)$ G.M.'s are inserted between 4 and 2916. Then the $(n + 1)^{th}$ G.M. is equal to
(a) 36 (b) 54
(c) 108 (d) 324

31. The sum of all integers of the form n^3 which lie between 100 and 10000 is
(a) 53361 (b) 53261
(c) 53214 (d) 53321

32. If g_1, g_2 are two geometric means, and a_1 is the arithmetic mean between two positive numbers, then the value of $\frac{g_1^2}{g_2}+\frac{g_2^2}{g_1}$ is
(a) $2a_1$ (b) a_1
(c) $\frac{a_1}{2}$ (d) $3a_1$

33. If x, y, z are in A.P., then
$\frac{1}{\sqrt{y}+\sqrt{z}}, \frac{1}{\sqrt{z}+\sqrt{x}}, \frac{1}{\sqrt{x}+\sqrt{y}}$ are in
(a) A.P. (b) G.P.
(c) A.G.P. (d) no definite sequence

34. If the n^{th} term of an A.P. is $5n^2 + 6n$ and r^{th} term is 401, then the value of r is
(a) 40 (b) 30
(c) 20 (d) 10

35. If $\Delta_r = \begin{vmatrix} 1 & n & n \\ 2r & n^2+n+1 & n^2+n \\ 2r-1 & n^2 & n^2+n+1 \end{vmatrix}$ and
$\sum_{r=1}^{n} \Delta_r = 90$, then the value of n is
(a) 9 (b) 10
(c) 6 (d) 8

36. The eighth term of an A.P., whose sum upto n terms is $2n^2 + n$, is
(a) 136 (b) 31
(c) 78 (d) 80

37. If a term of an infinite geometric series is thrice the sum of all the terms that follows it, then the common ratio is
(a) $\frac{1}{3}$ (b) $\frac{1}{2}$
(c) $\frac{1}{4}$ (d) $\frac{2}{3}$

38. The maximum value of the sum of the A.P. 50, 48, 46, 44, ..., is
(a) 325 (b) 648
(c) 650 (d) 652

39. The sum of the first hundred terms of an A.P. is x and the sum of the hundred terms starting from the third term is y. Then the common difference, is
(a) $\frac{y-x}{2}$ (b) $\frac{y-x}{50}$
(c) $\frac{y-x}{100}$ (d) $\frac{y-x}{200}$

40. The sum of the integers lying between 1 and 100 (both inclusive) and divisible by 3, 5, or 7, is
(a) 2838 (b) 3468
(c) 2738 (d) 3368

41. The value of the expression
$\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+\ldots+\frac{1}{20^2-1}$ is
(a) $\frac{11}{21}$ (b) $\frac{10}{21}$
(c) $\frac{10}{19}$ (d) $\frac{2}{19}$

42. The sum of 20 terms of the series
$1 + (1 + 3) + (1 + 3 + 5) + (1 + 3 + 5 + 7) + \ldots$.is
(a) 400 (b) 2870
(c) 5740 (d) 1540

43. If $1 \cdot 3 + 3 \cdot 3^2 + 5 \cdot 3^3 + 7 \cdot 3^4 + \ldots$ upto n terms is equal to $3 + (n - 1) \cdot 3^b$, then $b =$
(a) n (b) $n - 1$
(c) $2n - 1$ (d) $n + 1$

44. Let $p, q, r \in R$ and $27\,pqr \geq (p + q + r)^3$ and $3p + 4q + 5r = 12$ then $p^3 + q^4 + r^5$ is equal to
(a) 3 (b) 6
(c) 2 (d) none of these

45. If $\log 2$, $\log(2^x - 1)$ and $\log(2^x + 3)$ are in A.P., then the value of x is
(a) 5/2 (b) $\log_2 5$
(c) $\log_3 5$ (d) $\log_5 3$

46. The sum of the first n terms of the series $1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \ldots$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, the sum is
(a) $\frac{n^2(n+1)}{2}$ (b) $\frac{n(n+1)(2n+1)}{6}$
(c) $\frac{n(n+1)^2}{2}$ (d) $\frac{n^2(n+1)^2}{2}$

47. The minimum value of $8^{\sin(x/8)} + 8^{\cos(x/8)}$ is

(a) $2^{\frac{\frac{1}{3-\sqrt{2}}}{\sqrt{2}}}$ (b) $2^{\frac{3+\sqrt{2}}{\sqrt{2}}}$

(c) $2^{\frac{\frac{1}{3+\sqrt{2}}}{\sqrt{2}}}$ (d) $2^{\frac{3-\sqrt{2}}{\sqrt{2}}}$

48. The A.M. of the roots of a quadratic equation is A and G.M. of its roots is G. The quadratic equation is

(a) $x^2 + Ax + G^2 = 0$ (b) $x^2 + 2Ax + G^2 = 0$

(c) $x^2 - Ax + G^2 = 0$ (d) $x^2 - 2Ax + G^2 = 0$

49. Let S be the sum, P be the product and R be the sum of reciprocal of n terms of a G.P. Then

(a) $R = S \cdot P^{1/n}$ (b) $R = S \cdot P^{2/n}$

(c) $S = R \cdot P^{1/n}$ (d) $S = R \cdot P^{2/n}$

50. The ratio of the sum of first 3 terms to the sum of first 6 terms of a G.P. is 125 : 152. The common ratio of the G.P. is

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{3}{5}$ (d) $\frac{2}{5}$

51. The sum to 20 terms of the series $1 \times 3^2 + 2 \times 5^2 + 3 \times 7^2 + \ldots\ldots$, is

(a) 18800 (b) 188010

(c) 188020 (d) 188090

52. $\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots$ to 16 terms =

(a) 420 (b) 416 (c) 436 (d) 446

53. The sum of the first n terms of the series $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \ldots$ is

(a) $2^n - n + 1$ (b) $1 - 2^{-n}$

(c) $n - 1 + 2^{-n}$ (d) $2^n - 1$

54. If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)\, p^2 - 2(ab + bc + cd)p + b^2 + c^2 + d^2 \le 0$.Then a, b, c, d are

(a) in A.P. (b) in G.P.

(c) in A.G.P. (d) satisfy $ab = cd$

55. If the function f satisfies the relation $f(x + y) = f(x) \cdot f(y)$ for all natural numbers x, y, $f(1) = 2$ and $\sum_{r=1}^{n} f(a+r) = 16\,(2^n - 1)$, then the natural number a, is

(a) 2 (b) 3 (c) 4 (d) 5

56. Let x be the arithmetic mean and y, z be the two geometrical means between any two positive numbers. The value of $\frac{y^3 + z^3}{xyz}$ is

(a) 1 (b) 2 (c) 3 (d) 4

57. Let T_r be the r^{th} term of an A.P. for $r = 1, 2, 3, \ldots\ldots$ If for some positive integers m and n we have $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then $T_{mn} =$

(a) $\frac{1}{mn}$ (b) $\frac{1}{m} + \frac{1}{n}$

(c) 1 (d) 0

58. Let the positive numbers a, b, c, d be in A.P. Then $\frac{1}{abc}, \frac{1}{abd}, \frac{1}{acd}, \frac{1}{bcd}$ are in

(a) A.P. (b) G.P.

(c) A.G.P. (d) none of these

59. If x and y are positive real numbers and m, n are positive integers, then the minimum value of $\frac{x^m y^n}{(1 + x^{2m})(1 + y^{2n})}$ is

(a) 2 (b) $\frac{1}{4}$

(c) $\frac{1}{2}$ (d) 1

60. $1^3 - 2^3 + 3^3 - 4^3 + \ldots\ldots + 9^3 =$

(a) 425 (b) $-$ 425

(c) 475 (d) $-$ 475

61. $11^3 - 10^3 + 9^3 - 8^3 + 7^3 - 6^3 + 5^3 - 4^3 + 3^3 - 2^3 + 1^3 =$

(a) 756 (b) 724

(c) 648 (d) 812

62. If $\frac{bc}{ad} = \frac{b+c}{a+d} = 3\left(\frac{b-c}{a-d}\right)$, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in

(a) A.P. (b) G.P

(c) A.G.P. (d) none of these

63. If S_1, S_2, S_3 are the sums of n, $2n$, $3n$ terms respectively of an A.P., then $S_3/(S_2 - S_1) =$

(a) 1 (b) 2

(c) 3 (d) 4

64. The sum of four numbers in A.P. is 48, and the product of the extremes is to the product of the two middle terms as 27 : 35. The largest term of the A.P. is

(a) 10 (b) 12

(c) 14 (d) 18

65. If $a_1, a_2, \ldots, a_{n+1}$ are in A.P., with common difference d, then $\sum_{r=1}^{n} \tan^{-1} \frac{d}{1 + a_r a_{r+1}} =$

(a) $\tan^{-1}\dfrac{nd}{1+a_1a_{n+1}}$ (b) $\tan^{-1}\dfrac{(n+1)d}{1+a_1a_{n+1}}$

(c) $\tan^{-1}\dfrac{(n-1)d}{1-a_1a_{n+1}}$ (d) $\tan^{-1}\dfrac{(n+1)d}{1-a_1a_{n+1}}$

66. The A.M. between m and n and the G.M. between a and b are each equal to $\dfrac{ma+nb}{m+n}$ Then m =

(a) $\dfrac{a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ (b) $\dfrac{b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$

(c) $\dfrac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ (d) $\dfrac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$

67. 28 is divided into 4 parts which are in A.P. The ratio of the product of the first and third to the product of the second and fourth is 8 : 15. The largest part is

(a) 6 (b) 8

(c) 10 (d) 12

68. If 2, 7, 9, 5 are subtracted respectively from four numbers forming a G.P., the resulting numbers are in A.P., then the smallest of the four numbers is

(a) –24 (b) –12

(c) 6 (d) 3

69. The sum of the infinite series

$\dfrac{5}{13}+\dfrac{55}{13^2}+\dfrac{555}{13^3}+\ldots$ is

(a) $\dfrac{31}{18}$ (b) $\dfrac{65}{32}$

(c) $\dfrac{65}{36}$ (d) $\dfrac{75}{36}$

70. If $\dfrac{a+b}{1-ab}, b, \dfrac{b+c}{1-bc}$ are in A.P. then $\dfrac{1}{a}$, b, $\dfrac{1}{c}$ are in

(a) A.P. (b) G.P.

(c) A.G.P. (d) none of these

71. The coefficient of x^8 in the polynomial $(x - 1)(x - 2)(x - 3) \ldots\ldots (x - 10)$ is

(a) 1025 (b) 1240

(c) 1320 (d) 1440

72. If $a + c \neq b$ and $\dfrac{1}{a}+\dfrac{1}{c}+\dfrac{1}{a-b}+\dfrac{1}{c-b} = 0$ then $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

73. Sum to 10 terms of the series $1 + 2(1.1) + 3(1.1)^2 + 4(1.1)^3 + \ldots$, is

(a) 85. 12 (b) 92.5

(c) 96. 75 (d) none of these

74. The sum of the first 10 terms common to the series 17, 21, 25, ... and 16, 21, 26, ... is

(a) 1100 (b) 1010

(c) 1110 (d) 1200

75. If $S_1, S_2, S_3, \ldots, S_n$ are the sums of infinite geometric series whose first terms are 1, 2, 3, ..., n and common ratios $\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots, \dfrac{1}{n+1}$, then $S_1 + S_2 + S_3 + \ldots + S_n$ =

(a) $\dfrac{n(n+1)}{2}$ (b) $\dfrac{(n+1)(n+3)}{2}$

(c) $\dfrac{n(n+2)}{2}$ (d) $\dfrac{n(n+3)}{2}$

76. If a, b, c, d are in G.P., then $(b - c)^2 + (c - a)^2 + (b - d)^2$ =

(a) $(c + d)^2$ (b) $(c - d)^2$

(c) $(a + d)^2$ (d) $(a - d)^2$

77. If a, b, c, d are in G.P., then

$\dfrac{(a^2+b^2+c^2)(b^2+c^2+d^2)}{(ab+bc+cd)^2}$ =

(a) 1 (b) 2

(c) 3 (d) none of these

78. If $(m + 1)^{th}$, $(n + 1)^{th}$, $(r + 1)^{th}$ terms of the A.P. $a, a + d, a + 2d, \ldots$.. are in G.P. while $\dfrac{1}{m}, \dfrac{1}{n}, \dfrac{1}{r}$ are in A.P., then $\dfrac{d}{a}$ =

(a) $\dfrac{1}{n}$ (b) $-\dfrac{1}{n}$

(c) $-\dfrac{2}{n}$ (d) $\dfrac{2}{n}$

79. There are m A.Ms between 1 and 31. If the ratio of the 7^{th} and $(m - 1)^{th}$ means is 5 : 9, then m =

(a) 10 (b) 12

(c) 14 (d) 16

80. If one G.M. is g and two A.M.s are p and q, are inserted between two numbers a and b, then $\dfrac{(2p-q)(p-2q)}{g^2}$ =

(a) 1 (b) –1

(c) 2 (d) –3

81. If $x = 111....1$ (20 digits), $y = 333...3$ (10 digits) and $z = 222....2$ (10 digits), then $\frac{x-y^2}{z} =$

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) 3

82. The sum of the infinite A.G.P. 3, 4, 4...., is

(a) 27 (b) 30

(c) 24 (d) 25

83. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. The common ratio of the G.P., is

(a) $\frac{\sqrt{5}}{2}$ (b) $\sqrt{5}$

(c) $\frac{\sqrt{5}-1}{2}$ (d) $\frac{1-\sqrt{5}}{2}$

Assertion & Reason

Directions: *Each of these questions contains Statement-1 (Assertion) and Statement-2 (Reason). Each question has four choices. You have to select the correct choice.*

(a) if both statement-1 and statement-2 are true and statement-2 is the correct explanation of statement-1.

(b) if both statement-1 and statement-2 are true but statement-2 is not the correct explanation of statement-1.

(c) if statement-1 is true but statement-2 is false.

(d) if statement-1 is false and statement-2 is true.

1. Statement-1 : If the angles of a convex polygon are in A.P., 120°, 125°, 130°, ..., then it has 16 sides.

Statement-2 : The sum of the angles of a polygon of n sides is $(n - 2)180°$.

2. Statement-1 : If the infinite A.G.P. $1, \sqrt{3}, 2, x, ...$ has a finite sum, then $x = 2$

Statement-2 : The infinite A.G.P. $a, (a + d)r, (a + 2d)r^2, ...$ has a finite sum only if $|r| < 1$.

3. Statement-1 : There doesn't exist an A.P. whose three terms are $\sqrt{2}, \sqrt{3}, \sqrt{5}$.

Statement-2 : There exists distinct real numbers l, m, n such that

$\sqrt{2} = a+(l-1)d, \sqrt{3} = a+(m-1)d$ and $\sqrt{5} = a+(n-1)d$.

4. Statement-1 : If $(1^2 - a_1) + (2^2 - a_2) + ... + (n^2 - a_n) = \frac{1}{3}n(n^2-1)$, then $t_n = n + 1$.

Statement-2 : $1^2+2^2+........+n^2 = \frac{n(n+1)(2n+1)}{6}$.

5. Statement-1 : If three positive numbers in G.P. represent the sides of a triangle then the common ratio of the G.P. must lie between $\frac{\sqrt{5}-1}{2}$ and $\frac{\sqrt{5}+1}{2}$.

Statement-2 : Three positive numbers can form sides of a triangle if sum of any two sides is greater than the third side.

6. Statement-1 : If sum of n terms of two arithmetic progressions are in the ratio $(3n + 8) : (7n + 15)$, then ratio of their n^{th} terms is 3 : 16.

Statement-2 : If S_n is quadratic expression, then $t_n = S_n - S_{n-1}$.

7. Statement-1 : If a, b, c are distinct positive real numbers such that $a^2 + b^2 + c^2 = 1$, then $\Sigma ab = 2$.

Statement-2 : A.M. ≥ G.M.

8. Statement-1 : If S_n denotes the sum of n terms of a series given by $S_n = \frac{n(n+1)(n+2)}{6} \forall n \geq 1$, then $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{t_r} = 4$.

Statement-2 : $t_n = S_n - S_{n-1}$.

9. Statement-1 : For $n \in N, (n!)^3 < n^n\left(\frac{n+1}{2}\right)^{2n}$.

Statement-2 : $n > 6, \left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n$.

10. Statement-1 : If the sum of n terms of a series is $2n^2 + 3n + 1$, then series is in A.P. with common difference 4.

Statement-2 : If sum of n terms of an A.P. is quadratic expression, then common difference is twice of the coefficient of quadratic term.

11. Statement-1 : Let $2^{1/4}.4^{1/8}.8^{1/16}.16^{1/32} ... \infty = 2^x$, then the value of x is equal to 1.

Statement-2 : The sum of series

$\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+......\infty$ is equal to 1.

12. Statement-1 : There exist an A.P. with three terms $\sqrt{2}, \sqrt{3}, \sqrt{5}$.

Statement-2 : There exist three distinct real numbers a, b, c such that $\sqrt{2} = A+(a-1)d$, $\sqrt{3} = A+(b-1)d$, $\sqrt{5} = A+(c-1)d$.

13. Statement-1 : If all the terms of a series with positive terms are less than 10^{-8}, then sum of the infinite terms of the series is finite quantity.

Statement-2 : If $S_n < \frac{n}{10^8}$, then limiting value of S_n for $n \to \infty$ is not a finite quantity.

QUESTIONS FROM PREVIOUS YEARS AIEEE/JEE MAIN

1. $1^3 - 2^3 + 3^3 - 4^3 + ... + 9^3$ equals
(a) 425 (b) 400
(c) 405 (d) 395 **(2002)**

2. Sum of infinite terms of G.P is 20 and sum of their squares is 100, then common ratio of G.P. is
(a) 5 (b) $\frac{3}{5}$
(c) $\frac{8}{5}$ (d) $\frac{1}{5}$ **(2002)**

3. The value of $2^{\frac{1}{4}} \cdot 4^{\frac{1}{8}}\ 8^{\frac{1}{16}} \cdot 16^{\frac{1}{32}}$ ....∞ equals
(a) 1 (b) 2
(c) $\frac{3}{2}$ (d) 4 **(2002)**

4. Fifth term of G.P. is 2, then the product of its nine term is
(a) 256 (b) 512
(c) 1024 (d) none of these **(2002)**

5. Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and $a, b, c \in$ A.P. then $f'(a), f'(b), f'(c)$ are in
(a) G.P. (b) H.P.
(c) A.G.P. (d) A.P. **(2003)**

6. Sum of the series $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ...\infty$ is equal to
(a) $\log_e 2 - 1$ (b) $\log_e 2$
(c) $\log_e(4/e)$ (d) $2 \log_e 2$ **(2003)**

7. The real number x when added to its inverse gives the minimum value of the sum at x equal to
(a) 1 (b) –1
(c) –2 (d) 2 **(2003)**

8. If $a_1, a_2, a_3, ..., a_n \in$ G.P., then the value of the determinant

$$\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix} \text{ equals}$$

(a) 2 (b) 1
(c) 0 (d) –2 **(2004)**

9. Let T_r be the r^{th} term of an A.P. whose first term is 'a' and common difference is d. If for some positive integers $m, n, m \neq n$, $T_m = \frac{1}{n}$, $T_n = \frac{1}{m}$, then $a - d$ equals
(a) $\frac{1}{mn}$ (b) 1
(c) 0 (d) $\frac{1}{m}+\frac{1}{n}$ **(2004)**

10. The sum of n terms of the series (when n is even) $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + ...$, is $\frac{n(n+1)^2}{2}$. If n is odd, then sum will be
(a) $\frac{n(n+1)^2}{4}$ (b) $\frac{n^2(n+1)}{2}$
(c) $\frac{3n(n+1)}{2}$ (d) $\frac{n^2(n+1)^2}{4}$ **(2004)**

11. If $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$ where a, b, c are in A.P. and $|a| < 1$, $|b| < 1$, $|c| < 1$, then x, y, z are in
(a) H.P.
(b) Arithmetic-Geometric progression
(c) A.P. (d) G.P. **(2005)**

12. If in a ΔABC, the altitudes from the vertices A, B, C on opposite sides are in H.P., then $\sin A$, $\sin B$, $\sin C$ are in
(a) H.P.
(b) Arithmetic-Geometric progression
(c) A.P. (d) G.P. **(2005)**

13. Let $a_1, a_2, a_3, ...$ be terms of an A.P. If $\frac{a_1 + a_2 + ... + a_p}{a_1 + a_2 + ... + a_q} = \frac{p^2}{q^2}$, $p \neq q$, then $\frac{a_6}{a_{21}}$ equals
(a) 41/11 (b) 7/2
(c) 2/7 (d) 11/41 **(2006)**

14. If $a_1, a_2, ..., a_n$ are in H.P., then the expression $a_1a_2 + a_2a_3 + ... + a_{n-1}a_n$ is equal to
(a) $n(a_1 - a_n)$ (b) $(n - 1)(a_1 - a_n)$
(c) na_1a_n (d) $(n - 1)a_1a_n$ **(2006)**

15. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then the common ratio of this progression is equals

(a) $\sqrt{5}$ (b) $\frac{1}{2}(\sqrt{5}-1)$

(c) $\frac{1}{2}(1-\sqrt{5})$ (d) $\frac{1}{2}\sqrt{5}$. **(2007)**

16. The first two terms of a Geometric progression add up to 12. The sum of the third & fourth terms is 48. If the terms of G.P. are alternatively positive & negative then the first term is

(a) 4 (b) -12

(c) 12 (d) -4 **(2008)**

17. The sum to infinity of the series

$1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+......$ is

(a) 3 (b) 4

(c) 6 (d) 2 **(2009)**

18. A person is to count 4500 currency notes. Let a_n denote the number of notes he counts in the n^{th} minute. If $a_1 = a_2 = ... = a_{10} = 150$ and a_{10}, a_{11}, ... are in an A.P. with common difference -2, then the time taken by him to count all notes is

(a) 24 minutes (b) 34 minutes

(c) 125 minutes (d) 135 minutes

(2010)

19. A man saves ₹ 200 in each of the first three months of his service. In each of the subsequent months his saving increases by ₹ 40 more than the saving of immediately previous month. His total saving from the start of service will be ₹ 11040 after

(a) 20 months (b) 21 months

(c) 18 months (d) 19 months **(2011)**

20. **Statement 1 :** The sum of the series 1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + ... + (361 + 380 + 400) is 8000.

Statement 2 : $\sum_{k=1}^{n}(k^3-(k-1)^3)=n^3$, for any natural number n.

(a) Statement 1 is true, Statement 2 is true; Statement 2 is not a correct explanation for Statement 1.

(b) Statement 1 is true, Statement 2 is false.

(c) Statement 1 is false, Statement 2 is true.

(d) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1. **(2012)**

21. If 100 times the 100^{th} term of an A.P. with non-zero common difference equals the 50 times its 50^{th} term, then the 150^{th} term of this A.P. is

(a) 150 (b) zero

(c) – 150

(d) 150 times its 50^{th} term **(2012)**

22. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777, ..., is

(a) $\frac{7}{9}(99-10^{-20})$ (b) $\frac{7}{81}(179+10^{-20})$

(c) $\frac{7}{9}(99+10^{-20})$ (d) $\frac{7}{81}(179-10^{-20})$

(JEE Main 2013)

23. Three positive numbers form an increasing G.P. If the middle term in this G.P. is doubled, the new numbers are in A.P. Then the common ratio of the G.P. is

(a) $3+\sqrt{2}$ (b) $2-\sqrt{3}$

(c) $2+\sqrt{3}$ (d) $\sqrt{2}+\sqrt{3}$

(JEE Main 2014)

24. If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + ... + 10(11)^9 = k(10)^9$, then k is equal to

(a) $\frac{441}{100}$ (b) 100

(c) 110 (d) $\frac{121}{100}$

(JEE Main 2014)

25. The sum of first 9 terms of the series

$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+....$ is

(a) 142 (b) 192

(c) 71 (d) 96

(JEE Main 2015)

26. If m is A.M. of two distinct real numbers l and $n(l, n > 1)$ and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4+2G_2^4+G_3^4$ equals

(a) $4\ lmn^2$ (b) $4l^2m^2n^2$

(c) $4\ l^2mn$ (d) $4\ lm^2n$

(JEE Main 2015)

☺☺☺

HINTS & SOLUTIONS

Multiple Choice Questions

1. **(a) :** $a + b + c = xb$. Divide by b, $\frac{a}{b}+1+\frac{c}{b}=x$

$\frac{1}{r} + 1 + r = x$, r is the common ratio of the G.P.

$r^2 + r(1 - x) + 1 = 0$, r is real,

Discriminant > 0

$(1 - x)^2 - 4 > 0 \Rightarrow x^2 - 2x + 1 - 4 > 0$

$\Rightarrow x^2 - 2x - 3 > 0 \Rightarrow (x + 1)(x - 3) > 0$

$\Rightarrow x < -1$ or $x > 3$

2. **(a) :** $S = \frac{4}{19}+\frac{44}{19^2}+\frac{444}{19^3} + \ldots.\infty$ ......(1)

$\Rightarrow \frac{S}{19}=\frac{4}{19^2}+\frac{44}{19^3}+\ldots.\infty$ ......(2)

Subtracting (2) from (1), we get

$S.\frac{18}{19}=\frac{4}{19}+\frac{40}{19^2}+\frac{400}{19^3}+\ldots..$

$= \frac{4}{19}\left[1+\frac{10}{19}+\left(\frac{10}{19}\right)^2+\ldots..\right]=\frac{4}{19}\left[\frac{1}{1-\frac{10}{19}}\right]=\frac{4}{9}$

$\Rightarrow S=\frac{76}{162}=\frac{38}{81}.$

3. **(c) :** $\frac{a+2a}{2}=\sqrt{2a^2}+2 \Rightarrow (3a-4)^2=(2\sqrt{2}a)^2$

$\Rightarrow \frac{3a}{2}=\sqrt{2}a+2 \Rightarrow a^2-24a+16=0$

$\Rightarrow a=4\left(3\pm2\sqrt{2}\right)$

4. **(b) :** The three numbers are $\log_9 9$, $\log_{9^2}(3^x + 48)$ and $\log_9\left(3^x-\frac{8}{3}\right)$,

i.e. $\log_9 9$, $\frac{1}{2}\log_9(3^x+48)$, $\log_9\left(3^x-\frac{8}{3}\right)$ are in A.P.

$\Rightarrow \left\{(3^x+48)^{\frac{1}{2}}\right\}^2=9\left(3^x-\frac{8}{3}\right) \Rightarrow 8\cdot3^x = 72$

$\Rightarrow 3^x = 9 \Rightarrow x = 2.$

5. **(c) :** Expression

$= 2\log_2 a + 4\log_2 a + 6\log_2 a + \ldots.. + 40\log_2 a$

$= \log_2 a\ \{2 + 4 + \ldots. + 40\} = \frac{20}{2}(4 + 38)\log_2 a$

$= 420\log_2 a = 840 \Rightarrow \log_2 a = 2 \Rightarrow a = 4.$

6. **(a) :** $72 = 2^3 \times 3^2$; number of divisors of 72 are 12

$2025 = 3^4 \times 5^2$; number of divisors of 2025 are 15

$1568 = 2^5 \times 7^2$; number of divisors of 1568 are 18

Now 12, 15, 18 are in A.P.

7. **(c) :** Let $S = 1+2\left(1-\frac{1}{n}\right)+3\left(1-\frac{1}{n}\right)^2+\ldots..$ ...(1)

$\therefore \left(1-\frac{1}{n}\right)S=\left(1-\frac{1}{n}\right)+2\left(1-\frac{1}{n}\right)^2+\ldots..$ ...(2)

(1) — (2) gives $\frac{S}{n}=1+\left(1-\frac{1}{n}\right)+\left(1-\frac{1}{n}\right)^2+\ldots..\infty$

$=\frac{1}{1-\left(1-\frac{1}{n}\right)}=n \Rightarrow S = n^2$

8. **(d) :** Given $a_1 > 0$ and a_1, a_2, a_3 are in G.P. such that $a_2 = a_1r$ and $a_3 = a_1r^2$.

$\therefore 9a_1 + 5a_3 > 14a_2 \Rightarrow 9a_1 + 5a_1r^2 > 14a_1r$ and since $a_1 > 0$

we get $9 + 5r^2 > 14r \Rightarrow 5r^2 - 14r + 9 > 0$

$\Rightarrow (5r - 9)(r - 1) > 0 \Rightarrow r \notin \left[1,\frac{9}{5}\right]$

9. **(b) :** The given condition

$\Rightarrow (a_1b - a_2)^2 + (a_2b - a_3)^2 + (a_3b - a_4)^2 \le 0$

$\Rightarrow$ only the equality holds goods

$\Rightarrow a_1b = a_2,\ a_2b = a_3,\ a_3b = a_4$

$\Rightarrow b = \frac{a_2}{a_1}=\frac{a_3}{a_2}=\frac{a_4}{a_3}$

$\Rightarrow a_1, a_2, a_3, a_4$ are in G.P.

10. **(b) :** Given $a + c = 2b$.

Also $\frac{a+b+c}{3} \ge \sqrt[3]{abc}=\sqrt[3]{64}=4$

$\Rightarrow \frac{3b}{3} \ge 4 \Rightarrow b \ge 4 \Rightarrow$ minimum $b = 4$

11. **(d) :** We observe that all the terms of $\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}$ are contained in

$\left(1+\frac{1}{3}+\frac{1}{3^2}+\ldots..\right)\left(1+\frac{1}{5}+\frac{1}{5^2}+\ldots..\right).$

$\Rightarrow \frac{1}{a_1}+\frac{1}{a_2}+\ldots..+\frac{1}{a_n}<\left(\frac{1}{1-\frac{1}{3}}\right)\left(\frac{1}{1-\frac{1}{5}}\right)=\frac{3}{2}.\frac{5}{4}<2$

12. (b) : Given: $\cos^2\theta = \sin\theta\tan\theta = \dfrac{\sin^2\theta}{\cos\theta}$

$\Rightarrow \cos^3\theta = \sin^2\theta$ ...(1)

(1) $\Rightarrow \cos^6\theta = \sin^4\theta \Rightarrow \cot^6\theta = \dfrac{1}{\sin^2\theta}$

$\Rightarrow \cot^6\theta - \cot^2\theta = \dfrac{1}{\sin^2\theta} - \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{1-\cos^2\theta}{\sin^2\theta} = 1$

13. (b) : $S_n = \dfrac{3}{1^2} + \dfrac{5}{1^2+2^2} + \dfrac{7}{1^2+2^2+3^2} + \ldots$

$t_n = \dfrac{(2n+1)}{1^2+2^2+3^2+\ldots+n^2} = \dfrac{6(2n+1)}{n(n+1)(2n+1)} = \dfrac{6}{n(n+1)}$

$= 6\left[\dfrac{(n+1)-n}{n(n+1)}\right] = 6\left[\dfrac{1}{n} - \dfrac{1}{n+1}\right]$

$S_n = t_1 + t_2 + \ldots + t_n$

$= 6\left[\left(1-\dfrac{1}{2}\right) + \left(\dfrac{1}{2}-\dfrac{1}{3}\right) + \left(\dfrac{1}{3}-\dfrac{1}{4}\right) + \ldots + \left(\dfrac{1}{n}-\dfrac{1}{n+1}\right)\right]$

$= 6\left[1-\dfrac{1}{n+1}\right] = 6\left\{\dfrac{(n+1)-1}{(n+1)}\right\} = \dfrac{6n}{n+1}.$

14. (b) : $S = 2^{1/4}\cdot 4^{1/8}\cdot 8^{1/16}\cdot 16^{1/32}\ldots\infty$

$= 2^{1/4}\cdot 2^{2/8}\cdot 2^{3/16}\, 2^{4/32}\ldots$

$= 2^{1/4+2/8+3/16+4/32}\ldots = 2^x$

where $x = \dfrac{1}{4} + \dfrac{2}{8} + \dfrac{3}{16} + \dfrac{4}{32} + \ldots$ ...(1)

$\Rightarrow \dfrac{x}{2} = \dfrac{1}{8} + \dfrac{1}{8} + \dfrac{3}{32} + \dfrac{4}{64} + \ldots$ ...(2)

(1) – (2) $\Rightarrow \dfrac{x}{2} = \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \dfrac{1}{32} + \ldots = \dfrac{\frac{1}{4}}{1-\frac{1}{2}}$

$\Rightarrow x = 1 \Rightarrow S = 2$

15. (c) : $\sum_{j=1}^{21} a_j = 693 = \dfrac{21}{2}(a_1 + a_{21}) \quad \therefore a_1 + a_{21} = 66$

Now, a_{11} = A.M. = $\dfrac{693}{21} = 33$

Now, $a_2 + a_{20} = a_3 + a_{19} = \ldots$

$= a_9 + a_{13} = a_{10} + a_{12}$

$\therefore \sum_{i=0}^{10} a_{2i+1} = 5\times(a_1 + a_{21}) + a_{11} = 5 \times 66 + 33$
$= 363$

16. (b) : Let S_r be the sum of r terms of the given A.P. *i.e.*, $\dfrac{S_m}{S_n} = \dfrac{m^2}{n^2}$

$\Rightarrow \dfrac{S_m}{m^2} = \dfrac{S_n}{n^2} = \lambda$ (say).

If T_r is the r^{th} term, then $T_m = S_m - S_{m-1}$

$\Rightarrow T_m = \lambda(m^2 - (m-1)^2) = \lambda(2m-1)$. Similarly $T_n = \lambda(2n-1)$

$\therefore \; T_m : T_n = 2m - 1 : 2n - 1$

17. (c) : The given groups of numbers can be arranged as

$$\begin{array}{ccccccccc} & & & & 1 & & & & \to S_1 \\ & & 2 & & 3 & & 4 & & \to S_2 \\ 5 & & 6 & & 7 & & 8 & 9 & \to S_3 \\ & & \ldots\ldots & & & & & & \text{etc.,} \\ & & \ldots\ldots & & & & & & \\ & & \ldots\ldots & & & & & & \to S_{20} \end{array}$$

Number of terms upto S_{20} = 1 + 3 + 5 + ... to 20 terms = $\dfrac{20}{2}$ [2 + (19) 2] = 400

$\therefore S_{21}$ starts with 401.

18. (b) : Let $\dfrac{12}{r^2}, \dfrac{12}{r}, 12$ be a decreasing G.P.

$\therefore$ By data $\dfrac{12}{r^2}, \dfrac{12}{r}, 9$ are in A.P.

$\Rightarrow \dfrac{24}{r} = 9 + \dfrac{12}{r^2} \Rightarrow 3r^2 - 8r + 4 = 0$

$\Rightarrow r = 2$ or $\dfrac{2}{3} \Rightarrow r = \dfrac{2}{3}$ only

($\therefore$ The given G.P. is a decreasing one).

19. (b) : $T_r = r(n - (r-1)) = r(n - r + 1)$

$= nr - r^2 + r$

$\sum_{r=1}^{n} T_r = \sum_{r=1}^{n}(nr - r^2 + r) = (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2$

$= (n+1)\left[\dfrac{n(n+1)}{2}\right] - \dfrac{n(n+1)(2n+1)}{6}$

$= \dfrac{n(n+1)(n+2)}{6}$

20. (b) : $21 = 1^2 + 2^2 + 4^2$ so that

$(1^2 + 2^2 + 4^2)(x^2 + y^2 + z^2) = (x + 2y + 4z)^2$

i.e., $(2x - y)^2 + (4y - 2z)^2 + (z - 4x)^2 = 0$

or $x = \dfrac{y}{2} = \dfrac{z}{4}$, A.G.P. with C.R. = $\dfrac{1}{2}$.

21. (c) : The given series is

$\sum_{r=1}^{n} \dfrac{r^2+1}{r(r+1)} 2^{r-1} = \sum_{1}^{n}\left(\dfrac{2r}{r+1} - \dfrac{r-1}{r}\right)2^{r-1}$

$= \sum_{1}^{n}\left(\dfrac{r}{r+1}2^r - \dfrac{r-1}{r}2^{r-1}\right) = \dfrac{n}{n+1}2^n$

22. (a) : The given series

$= \sum_{r=1}^{n}(n-2r)^2 = \sum_{1}^{n}(n^2 - 4nr + 4r^2)$

$= n\cdot n^2 - 4n\cdot\dfrac{n(n+1)}{2} + 4.\dfrac{n}{6}(n+1)(2n+1)$

$= \dfrac{n}{3}(n^2+2)$

23. (c) : Three numbers in A.P. can be taken as $a - d, a, a + d$

Then $(a-d)^2, a^2, (a+d)^2$ are in G.P.

$\Rightarrow a^4 = (a^2-d^2)^2 \Rightarrow d^4 - 2a^2d^2 = 0$

$\Rightarrow d^2(d^2-2a^2) = 0 \Rightarrow d = 0, \pm\sqrt{2}a$

The common ratio $\left(\frac{a+d}{a}\right)^2$

when $d = 0, \left(\frac{a+d}{d}\right)^2 = 1$

when $d = \pm\sqrt{2}a, \left(\frac{d+a}{a}\right)^2 = \left(\frac{a\pm\sqrt{2}a}{a}\right)^2$

$= (1\pm\sqrt{2})^2 = 3\pm 2\sqrt{2}$

Thus there are three common ratios

$1, 3+2\sqrt{2}, 3-2\sqrt{2}$.

24. (c) : $x^{15} - x^{13} + x^{11} - x^9 + x^7 - x^5 + x^3 - x$

$= x(x^8+1)(x^4+1)(x^2-1)$

$x^{16} - 1 = (x^8+1)(x^4+1)(x^2+1)(x^2-1)$

$= \frac{7}{x}(x^2+1)$

$= 7\left[\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 + 2\right] > 14 \therefore x^{16} > 15$

25. (a) : If a is the first term and d is the common difference, $a + 5d = 18$ and $a + 8d = 12$ giving $d = -2$ and $a = 28$.

$\therefore 15^{\text{th}}$ term $a_{15} = 28 + 14(-2) = 0$.

26. (b) : $a + b = 3\cdot 2\sqrt{ab}$ or $\frac{a}{b} - 6\sqrt{\frac{a}{b}} + 1 = 0 \Rightarrow \sqrt{\frac{a}{b}}$

$= 3\pm 2\sqrt{2}$ or $\frac{a}{b} = \frac{3+2\sqrt{2}}{3-2\sqrt{2}}\cdot\frac{3-2\sqrt{2}}{3+2\sqrt{2}}$

As $a^2 + b^2 = 34$, the two numbers are $3+2\sqrt{2}$ and $3-2\sqrt{2}$.

27. (a) : Let $a, \underbrace{ar, ar^2, ar^3, ar^4}, ar^5$ be the 6 terms of the G.P. having 4 G.M.'s between a and ar^5. We have $a = 2^{11} - 1$ and $ar^5 = 2^{11} + 1$

Product of geometric means $= (ar)(ar^2)(ar^3)(ar^4)$

$= a^4r^{1+2+3+4} = a^4r^{10} = (a^2r^5)^2 = \{(a)(ar^5)\}^2$

$= \{(2^{11}-1)(2^{11}+1)\}^2 = (2^{22}-1)^2 = 2^{44} - 2^{23} + 1$.

28. (d) : $t_m = a + (m-1)d = \frac{1}{n}$ ......(1)

$t_n = a + (n-1)d = \frac{1}{m}$ .....(2)

Subtracting (2) from (1) we get

$(m-n)d = \frac{1}{n} - \frac{1}{m} \Rightarrow (m-n)d = \frac{m-n}{mn}$

$\therefore d = \frac{1}{m} (\because m \neq n)$ ....(3)

$t_{mn} = a + (mn-1)d = a + (mn-1)\times\frac{1}{mn}$

$= a - \frac{1}{mn} + 1$ ....(4)

From (1) and (3)

$a + (m-1)\cdot\frac{1}{mn} = \frac{1}{n} \Rightarrow a + \frac{1}{n} - \frac{1}{mn} = \frac{1}{n}$

$\Rightarrow a = \frac{1}{mn} \Rightarrow t_{mn} = 1$

29. (c) : $\frac{a+b}{2} + \sqrt{ab} = a - b \Rightarrow 2\sqrt{ab} = a - 3b$

$\Rightarrow 4ab = (a-3b)^2 \Rightarrow a^2 - 10ab + 9b^2 = 0$

$\Rightarrow (a-9b)(a-b) = 0$

$\Rightarrow a = 9b\ [\because a \neq b] \Rightarrow \frac{a}{b} = \frac{9}{1}$

30. (c) : The $(n+1)^{\text{th}}$ G.M. = middle G.M. = G.M. of 4 and 2916 $= \sqrt{4\times 2916} = 108$

31. (b) : The numbers are

$125 = 5^3, 6^3, 7^3, ..., 21^3 = 9261; 22^3 = 10648$

Required Sum $= (1^3 + 2^3 + ... + 21^3) - (1^3 + 2^3 + 3^3 + 4^3)$

$= \left(\frac{21\times 22}{2}\right)^2 - \left(\frac{4\times 5}{2}\right)^2 = 53361 - 100 = 53261$

32. (a) : Let a, g_1, g_2, b are in G.P.

$\therefore g_1^2 = ag_2 \Rightarrow \frac{g_1^2}{g_2} = a$

g_1, g_2, b are in G.P.

$\therefore g_2^2 = bg_1 \Rightarrow \frac{g_2^2}{g_1} = b$

$\Rightarrow \frac{g_1^2}{g_2} + \frac{g_2^2}{g_1} = a + b = 2a_1$

$[\because a, a_1, b$ are in A.P.$]$

33. (a) : $\frac{1}{\sqrt{z}+\sqrt{x}} - \frac{1}{\sqrt{y}+\sqrt{z}} = \frac{\sqrt{y}-\sqrt{x}}{(\sqrt{z}+\sqrt{x})(\sqrt{y}+\sqrt{z})}$

$= \frac{(\sqrt{y}+\sqrt{x})(\sqrt{y}-\sqrt{x})}{(\sqrt{x}+\sqrt{y})(\sqrt{z}+\sqrt{x})(\sqrt{y}+\sqrt{z})}$

$= \frac{y-x}{(\sqrt{x}+\sqrt{y})(\sqrt{z}+\sqrt{x})(\sqrt{y}+\sqrt{z})}$

Similarly

$\frac{1}{\sqrt{x}+\sqrt{y}} - \frac{1}{\sqrt{z}+\sqrt{x}} = \frac{z-y}{(\sqrt{x}+\sqrt{y})(\sqrt{z}+\sqrt{x})(\sqrt{y}+\sqrt{z})}$

x, y, z are in A.P. $\Rightarrow y - x = z - y \Rightarrow x + z = 2y$

$\Rightarrow$ Given numbers are in A.P.

34. (a) : $T_n = S_n - S_{n-1}$ *i.e.,*

$T_n = 5n^2 + 6n - 5(n-1)^2 - 6(n-1) = 10n + 1$

$\therefore\ T_r = 10r + 1 = 401$

$\Rightarrow 10r = 400 \ \Rightarrow\ r = 40$

35. (a) : $\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} n & n & n \\ n(n+1) & n^2+n+1 & n^2+n \\ n^2 & n^2 & n^2+n+1 \end{vmatrix}$

$= \begin{vmatrix} 0 & n & n \\ -1 & n^2+n+1 & n^2+n \\ 0 & n^2 & n^2+n+1 \end{vmatrix}\ C_1 \to C_1 - C_2$

$= \begin{vmatrix} n & n \\ n^2 & n^2+n+1 \end{vmatrix} = n^2 + n = n(n+1) = 90$

$\therefore n = 9$

36. (b) : $T_8 = S_8 - S_7 = (2 \cdot 8^2 + 8) - (2 \cdot 7^2 + 7) = 31$

37. (c) : Given

$ar^{n-1} = 3(ar^n + ar^{n+1} + ar^{n+2} + ...\infty)$

$= 3ar^n(1 + r + r^2 + ...)$

$= 3ar^n \dfrac{1}{1-r} \ \Rightarrow\ 1 - r = 3r \Rightarrow r = \dfrac{1}{4}$

38. (c) : The maximum value will correspond to n terms when the n^{th} term is either zero or the smallest positive number of the series.

i.e., $50 + (n - 1)(-2) = 0$ when $n = 26$;

$S_{26} = \dfrac{26}{2}(a + b) = 13\,(50 + 0) = 650$

39. (d) : $\dfrac{100}{2}\{2a + 99d\} = x$;

$\dfrac{100}{2}\{2(a + 2d) + 99d\} = y$

On subtraction, $200d = y - x \Rightarrow d = \dfrac{y-x}{200}$

40. (a) : The integers divisible by 3 are 33 in number and are 3, 6, ..., 99.

The integers divisible by 5 are 20 in number and are 5, 10, ..., 100.

The integers divisible by 7 are 14 in number and are 7, 14, ..., 98.

The integers divisible by both 3 and 5 are 6 in number and are 15, 30, ..., 90.

The integers divisible by both 3 and 7 are 4 in number and are 21, 42, 63 and 84.

The integers divisible by both 5 and 7 are 2 in number and are 35 and 70.

There are no integers divisible by all three.

Hence the sum of the numbers divisible by 3 or 5 or 7 is

$\dfrac{33}{2}(3 + 99) + \dfrac{20}{2}(5 + 100) + \dfrac{14}{2}(7 + 98) - \dfrac{6}{2}(15 + 90) - \dfrac{4}{2}(21 + 84) - (35 + 70) = 2838.$

41. (b) : $\dfrac{1}{(2n)^2-1} = \dfrac{1}{(2n-1)(2n+1)} = \dfrac{1}{2}\left[\dfrac{1}{(2n-1} - \dfrac{1}{2n+1}\right]$

Set $n = 1, 2, ..., 10$

$\dfrac{1}{2^2-1} = \dfrac{1}{2}\left[\dfrac{1}{1} - \dfrac{1}{3}\right]$

$\dfrac{1}{4^2-1} = \dfrac{1}{2}\left[\dfrac{1}{3} - \dfrac{1}{5}\right]$

....

$\dfrac{1}{20^2-1} = \dfrac{1}{2}\left[\dfrac{1}{19} - \dfrac{1}{21}\right]$

Adding we get $S = \dfrac{1}{2}\left(1 - \dfrac{1}{21}\right) = \dfrac{1}{2} \times \dfrac{20}{21} = \dfrac{10}{21}.$

42. (b) : $t_n = 1 + 3 + 5 + ... + (2n - 1) = n^2$

$S_{20} = 1^2 + 2^2 + 3^2 + ... + 20^2 = \dfrac{20 \times 21 \times 41}{6}$

$= 2870.$

43. (d) : $S = 1 \cdot 3 + 3 \cdot 3^2 + 5 \cdot 3^3 + 7 \cdot 3^4 + ...$

$.............................. + (2n - 1)3^n$

$3S = 1 \cdot 3^2 + 3 \cdot 3^3 + ... + (2n - 3)3^n + (2n - 1)3^{n+1}$

$-2S = 1 \cdot 3 + 2[3^2 + 3^3 + ... + 3^n] - (2n - 1)3^{n+1}$

$S = \left(\dfrac{2n-1}{2}\right)3^{n+1} - \dfrac{2 \cdot 3(3^n - 1)}{2(3-1)} + \dfrac{1 \cdot 3}{2}$

$= 3 + (n - 1)3^{n+1}$

But given $= 3 + (n - 1)3^b \ \therefore\ b = n + 1$

44. (a) : $27\,pqr \geq (p + q + r)^3 \Rightarrow (pqr)^{1/3} \geq \left(\dfrac{p+q+r}{3}\right)$

$\Rightarrow\ p = q = r.$ Also $3p + 4q + 5r = 12$

$\Rightarrow p = q = r = 1$

45. (b) : $2\log(2^x - 1) = \log 2 + \log(2^x + 3)$

$\Rightarrow\ (2^x - 1)^2 = 2 \cdot (2^x + 3)$

$\Rightarrow (2^x)^2 - 4 \cdot 2^x - 5 = 0$

$\Rightarrow\ (2^x - 5)(2^x + 1) = 0$

$\Rightarrow x = \log_2 5$, as $2^x + 1 \neq 0$

46. (a) : If n is odd, $n - 1$ is even. Sum of $(n - 1)$ terms will be $\dfrac{(n-1)(n-1+1)^2}{2} = \dfrac{n^2(n-1)}{2}.$

The n^{th} term will be n^2.

Hence the required sum

$= \dfrac{n^2(n-1)}{2} + n^2 = \dfrac{n^2(n+1)}{2}$

47. (b) : A.M. $\geq$ G.M.

$\Rightarrow \dfrac{2^{3\sin x/8} + 2^{3\cos x/8}}{2} \geq \sqrt{2^{3\sqrt{2}\cos\left(\frac{x}{8}-\frac{\pi}{4}\right)}}$

Now maximum of $\sqrt{2^{3\sqrt{2}\cos\left(\frac{x}{8}-\frac{\pi}{4}\right)}} = \sqrt{2^{3\sqrt{2}\cdot 1}} = 2^{\frac{3}{\sqrt{2}}}$

So A.M. $\geq 2^{3/\sqrt{2}} \ \Rightarrow\ 8^{\sin x/8} + 8^{\cos x/8} \geq 2^{\left(\frac{3}{\sqrt{2}}+1\right)}$

48. (d) : The equation will be

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ has roots α, β

$$A = \frac{\alpha+\beta}{2},\ G = \sqrt{\alpha\beta}$$

$\therefore\ \alpha + \beta = 2A,\ \alpha\beta = G^2$

The equation is $x^2 - 2Ax + G^2 = 0$.

49. (d): Let $a, ar, ar^2, \ldots\ldots$ be the G.P.

$$S = \frac{a(1-r^n)}{1-r}$$

The reciprocals are in G.P.

$$\frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}, \ldots..$$

$$R = \frac{\frac{1}{a}\left(1-\frac{1}{r^n}\right)}{1-\frac{1}{r}} = \frac{1}{a}\left(\frac{1-r^n}{1-r}\right)\frac{1}{r^{n-1}}$$

$$\therefore\ \frac{S}{R} = a^2 r^{n-1} \quad \ldots (1)$$

$P = a \cdot ar \cdot ar^2 \ldots ar^{n-1}$

$= a^n r^{(1+2+\ldots+(n-1))} = a^n r^{(n-1)n/2}$

$$\therefore\ P^2 = a^{2n} \cdot r^{n(n-1)} = (a^2 r^{n-1})^n = \left(\frac{S}{R}\right)^n \text{ (by (1))}$$

$\Rightarrow\ S = R \cdot P^{2/n}$.

50. (c): Let $a, ar, ar^2, \ldots..$ be the G.P.

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\frac{S_3}{S_6} = \frac{125}{152} \Rightarrow \frac{1-r^3}{1-r^6} = \frac{125}{152}$$

$$\frac{1}{1+r^3} = \frac{125}{152} \Rightarrow r = \frac{3}{5}$$

51. (d): The r^{th} term of the series

$t_r = r(2r+1)^2 = 4r^3 + 4r^2 + r$

$$\sum_{r=1}^{20} t_r = 4\sum_{r=1}^{20} r^3 + 4\sum_{r=1}^{20} r^2 + \sum_{r=1}^{20} r$$

$$= 4\cdot\left(\frac{20\times 21}{2}\right)^2 + \frac{4\cdot 20\cdot 21\cdot 41}{6} + \frac{20\cdot 21}{2}$$

$= 4(210)^2 + 40 \cdot 7.\ 41 + 210 = 188090$.

52. (d): The r^{th} term, $t_r = \dfrac{1^3 + 2^3 + \ldots + r^3}{1 + 3 + \ldots + (2r-1)}$

$$= \left(\frac{r(r+1)}{2}\right)^2 \cdot \frac{1}{r^2} = \frac{1}{4}(r+1)^2$$

$$\sum_{r=1}^{16} t_r = \frac{1}{4}[2^2 + 3^2 + \ldots.+17^2] = \frac{1}{4}\left[\frac{17\times 18\times 35}{6} - 1\right]$$

$= 446$.

53. (c): $\dfrac{1}{2} + \dfrac{3}{4} + \dfrac{7}{8} + \ldots$

$$= \left(1-\frac{1}{2}\right) + \left(1-\frac{1}{4}\right) + \left(1-\frac{1}{8}\right) + \ldots \text{ to } n \text{ terms}$$

$$= n - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \text{to } n \text{ terms}\right), \text{ a G.P.}$$

$$= n - \frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1/2} = n - 1 + 2^{-n}.$$

54. (b): $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + b^2 + c^2 + d^2 \le 0$

$\Rightarrow (ap - b)^2 + (bp - c)^2 + (cp - d)^2 \le 0$

The sum of squares cannot be negative.

$\therefore (ap - b)^2 + (bp - c)^2 + (cp - d)^2 = 0$

$ap - b = bp - c = cp - d = 0$

$p = \dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c} \Rightarrow a, b, c, d$ are in G.P.

55. (b): $f(2) = f(1+1) = f(1)\cdot f(1) = 2^2$

$f(3) = f(2+1) = f(2)\cdot f(1) = 2^2 \cdot 2 = 2^3$

$$f(n) = 2^n 16(2^n - 1) = \sum_{r=1}^{n} f(a+r) = \sum_{r=1}^{n} 2^{a+r}$$

$$= 2^a(2 + 2^2 + \ldots.. + 2^n) = 2^a \cdot 2\left(\frac{2^n - 1}{2-1}\right)$$

$= 2^{a+1}(2^n - 1)$, by the sum of G.P.

$\therefore\ 2^{a+1} = 16$ or $a = 3$.

56. (b): Let the two numbers be a and b.

$x = \text{A.M.} = \dfrac{a+b}{2}$

$a,\ y = ar,\ z = ar^2,\ b = ar^3$ are in G.P.

$$\frac{y^3 + z^3}{xyz} = \frac{a^3r^3 + a^3r^6}{\frac{(a+b)}{2}\cdot a^2r^3}$$

$$\frac{2a(1+r^3)}{a+ar^3} = 2.$$

57. (c): Let $a, a+d, a+2d, \ldots.$be the A.P.

$T_m = \dfrac{1}{n} \Rightarrow a + (m-1)d = \dfrac{1}{n}$

$T_n = \dfrac{1}{m} \Rightarrow a + (n-1)d = \dfrac{1}{m}$

Solving $a = d = \dfrac{1}{mn}$

$T_{mn} = a + (mn-1)d = \dfrac{1 + mn - 1}{mn} = 1$

58. (a): a, b, c, d are in A.P. Reversing the terms, d, c, b, a are in A.P.

Dividing each terms by $abcd$,

$\dfrac{1}{abc}, \dfrac{1}{abd}, \dfrac{1}{acd}, \dfrac{1}{bcd}$ are in A.P.

59. (b): A.M. ≥ G.M. $\Rightarrow \frac{1+x^{2m}}{2} \geq x^m$,

and $\frac{1+y^{2n}}{2} \geq y^n$

$\therefore \frac{x^m}{1+x^{2m}} \cdot \frac{y^n}{1+y^{2n}} \leq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

60. **(a):** $1^3 - 2^3 + 3^3 - 4^3 + \ldots\ldots + 9^3$
$= 1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots + 9^3 - 2(2^3 + 4^3 + 6^3 + 8^3)$

$= \left(\frac{9 \times 10}{2}\right)^2 - 16(1^3 + 2^3 + 3^3 + 4^3)$

$= 45^2 - 16\left(\frac{4 \times 5}{2}\right)^2 = 45^2 - 40^2 = 425.$

61. **(a):** $1^3 - 2^3 + 3^3 - \ldots + 11^3$
$= 1^3 + 2^3 + 3^3 + \ldots + 11^3 - 2(2^3 + 4^3 + 6^3 + 8^3 + 10^3)$

$= \left(\frac{11 \times 12}{2}\right)^2 - 16\,(1^3 + 2^3 + 3^3 + 4^3 + 5^3)$

$= 66^2 - 16 \cdot \left(\frac{5 \times 6}{2}\right)^2 = 66^2 - 60^2 = 756.$

62. **(a):** $\frac{bc}{ad} = \frac{b+c}{a+d} \Rightarrow \frac{b+c}{bc} = \frac{a+d}{ad}$

$\Rightarrow \frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{d}$ ...(1)

$\frac{bc}{ad} = 3\left(\frac{b-c}{a-d}\right) \Rightarrow \frac{a-d}{ad} = \frac{3(b-c)}{bc}$

$\Rightarrow \frac{1}{d} = \frac{1}{a} + 3\left(\frac{1}{c} - \frac{1}{b}\right)$ ... (2)

Let $\frac{1}{b} - \frac{1}{a} = \alpha$. Then (1) $\Rightarrow \frac{1}{d} - \frac{1}{c} = \alpha$,

Substituting for $\frac{1}{b}$ and $\frac{1}{d}$ in (2), we get

$\frac{1}{c} + \alpha = \frac{1}{a} + \frac{3}{c} - 3\left(\frac{1}{a} + \alpha\right)$

$\Rightarrow \frac{2}{c} = \frac{2}{a} + 4\alpha$

$\therefore \frac{1}{b} = \frac{1}{a} + \alpha, \frac{1}{c} = \frac{1}{a} + 2\alpha, \frac{1}{d} = \frac{1}{a} + 3\alpha$

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in A.P.

63. **(c):** $S_1 = \frac{n}{2}[2a + (n-1)d]$

$S_2 = n[2a + (2n-1)d]$

$S_2 - S_1 = na + (3n-1)\frac{nd}{2} = \frac{n}{2}[2a + (3n-1)d]$

$S_3 = \frac{3n}{2}[2a + (3n-1)d]$

$\therefore S_3/(S_2 - S_1) = 3.$

64. **(d):** Let the four numbers of A.P. be $a - 3d, a - d, a + d, a + 3d$.
The sum of the terms = 48 $\Rightarrow a = 12$

$\frac{(12-3d)(12+3d)}{(12-d)(12+d)} = \frac{27}{35}$

$\Rightarrow 35\,(16 - d^2) = 3\,(144 - d^2)$

$\Rightarrow d^2 = 4 \Rightarrow d = \pm 2$

The number are 12 – 6, 12 – 2, 12 + 2, 12 + 6 or 12 + 6, 12 + 2, 12 – 2, 12 – 6

$\therefore$ 6, 10, 14, 18 or 18, 14, 10, 6,

$\therefore$ largest term is 18.

65. **(a):** $\sum_{r=1}^{n} \tan^{-1}\frac{d}{1+a_r a_{r+1}} = \sum_{r=1}^{n} \tan^{-1}\frac{a_{r+1}-a_r}{1+a_r a_{r+1}}$

$= \sum_{r=1}^{n}(\tan^{-1} a_{r+1} - \tan^{-1} a_r)$

$= \tan^{-1} a_{n+1} - \tan^{-1} a_1$

$= \tan^{-1}\frac{a_{n+1}-a_1}{1+a_1 a_{n+1}} = \tan^{-1}\frac{nd}{1+a_1 a_{n+1}}$

66. **(d) :** $\frac{m+n}{2} = \sqrt{ab} = \frac{ma+nb}{m+n}$

$(m+n)\sqrt{ab} = ma + nb$

$m\left(\sqrt{ab} - a\right) = n\left(b - \sqrt{ab}\right)$

$m\left(\sqrt{b} - \sqrt{a}\right)\sqrt{a} = n\left(\sqrt{b} - \sqrt{a}\right)\sqrt{b}$

$\frac{m}{\sqrt{b}} = \frac{n}{\sqrt{a}} = k$, say ...(1)

Further $\frac{m+n}{2} = \sqrt{ab}$ ...(2)

(1), (2) $\Rightarrow \frac{k}{2}\left(\sqrt{b} + \sqrt{a}\right) = \sqrt{ab}$

$\therefore k = \frac{2\sqrt{ab}}{\sqrt{a}+\sqrt{b}} \Rightarrow m = k\sqrt{b} = \frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$.

67. **(c):** Since the sum is 28, we take the A.P. as $7 - 3d, 7 - d, 7 + d, 7 + 3d$.

$\frac{(7-3d)(7+d)}{(7-d)(7+3d)} = \frac{8}{15}$

$15[49 - 14d - 3d^2] = 8[49 + 14d - 3d^2]$

$\Rightarrow 3d^2 + 46d - 49 = 0 \Rightarrow d = 1$

$\therefore$ The parts are 7 – 3, 7 – 1, 7 + 1, 7 + 3 or 4, 6, 8, 10.

68. **(a):** Let the numbers be a, ar, ar^2, ar^3
$\Rightarrow a - 2, ar - 7, ar^2 - 9, ar^3 - 5$ are in A.P.

$a - 2 + ar^2 - 9 = 2(ar - 7)$

$\Rightarrow a(r^2 + 1) - 11 = 2ar - 14$

$\Rightarrow r^2 + 1 = 2r - \frac{3}{a}$ ...(1)

Further, $ar - 7 + ar^3 - 5 = 2\,(ar^2 - 9)$

$\Rightarrow ar(r^2 + 1) - 12 = 2ar^2 - 18$ ...(2)

$\Rightarrow r^2 + 1 = 2r - \frac{6}{ar}$

(1), (2) $\Rightarrow r = 2, a = -3$

The numbers are –3, – 6, –12, –24.

69. (c) : $S = \frac{5}{13} + \frac{55}{13^2} + \frac{555}{13^3} + ...$

$\frac{S}{13} = \frac{5}{13^2} + \frac{55}{13^3} + ...$

Subtracting,

$\frac{12}{13}S = \frac{5}{13} + \frac{50}{13^2} + \frac{500}{13^3}...$

Which is a G.P. with common ratio is $\frac{10}{13}$

$\therefore S = \frac{13}{12} \times \left[\frac{5}{13} \div \left(1 - \frac{10}{13}\right)\right] = \frac{65}{36}$

70. (a): $\frac{a+b}{1-ab} + \frac{b+c}{1-bc} = 2b$

$(a + b)(1 - bc) + (b + c)(1 - ab) = 2b(1 - ab)(1 - bc)$

$\Rightarrow (b^2 + 1)(2abc - (a + c)) = 0$ (on simplification).

$\therefore \frac{1}{b} = \frac{2ac}{a+c}$ (on simplification).

71. (c): The coefficient of x^8 is

$1 \cdot 2 + 1 \cdot 3 + ... + 2 \cdot 3 + 2 \cdot 4 + ...$

$= \frac{1}{2}\left[(1 + 2 + ...10)^2 - (1^2 + 2^2 + ... + 10^2)\right]$

$= \frac{1}{2}\left[\left(\frac{10 \times 11}{2}\right)^2 - \frac{10 \times 11 \times 21}{6}\right]$

$= \frac{1}{2}[55^2 - 55 \times 7] = 55 \times 24$

$= 1320.$

72. (a): $\left(\frac{1}{a} + \frac{1}{c-b}\right) + \left(\frac{1}{c} + \frac{1}{a-b}\right) = 0$

$\frac{(a+c-b)}{a(c-b)} + \frac{(a+c-b)}{c(a-b)} = 0$

$\Rightarrow \frac{1}{a(c-b)} + \frac{1}{c(a-b)} = 0$

$\Rightarrow c(a - b) + a(c - b) = 0 \Rightarrow b = \frac{2ca}{c+a}$.

73. (d): Let $x = 1.1$

$S = 1 + 2x + 3x^2 + ... + 10x^9$ ......(i)

$Sx = x + 2x^2 + ... + 9x^9 + 10x^{10}$ ......(ii)

Subtracting, (ii) from (i), we get

$S(1 - x) = 1 + x + x^2 + ... + x^9 - 10x^{10}$

$= \frac{x^{10}-1}{x-1} - 10x^{10} = \frac{x^{10}-1}{0.1} - 10x^{10}$ since $x = 1.1$

$= 10(x^{10} - 1) - 10x^{10} = -10$

$\therefore S = \frac{10}{x-1} = \frac{10}{0.1} = 100.$

74. (c): The first common term is 21. The common difference of the first series is 4.

The common difference of the second series is 5.

The common difference of the common series is the *L.C.M.* of 4 and 5, i.e. 20.

$\therefore$ The sum of first 10 common terms is

$\Rightarrow \frac{10}{2}[2 \times 21 + 9 \times 20] = 1110.$

75. (d): S_r is the sum of infinite G.P. with first term r and common ratio $\frac{1}{r+1}$

$\therefore S_r = \frac{r}{1 - \frac{1}{r+1}} = r + 1$

$\sum_{r=1}^{n} S_r = \sum_{r=1}^{n}(r+1) = \frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}$.

76. (d): Since $b^2 = ca$, $c^2 = bd$, $bc = ad$

$(b - c)^2 + (c - a)^2 + (b - d)^2 = b^2 - 2bc + c^2 + c^2 - 2ca + a^2 + b^2 - 2bd + d^2 = a^2 + d^2 - 2bc$

$= a^2 + d^2 - 2ad = (a - d)^2$.

77. (a): $b = ar$, $c = ar^2$, $d = ar^3$

$a^2 + b^2 + c^2 = a^2(1 + r^2 + r^4)$

$b^2 + c^2 + d^2 = a^2r^2(1 + r^2 + r^4)$

$ab + bc + cd = a^2 r(1 + r^2 + r^4)$

$\therefore \frac{(a^2+b^2+c^2)(b^2+c^2+d^2)}{(ab+bc+cd)^2} = 1.$

78. (c): $a + md$, $a + nd$, $a + rd$ are in G.P.

$\therefore (a + md)(a + rd) = (a + nd)^2$

$\Rightarrow (1 + m\lambda)(1 + r\lambda) = (1 + n\lambda)^2$, $\lambda = \frac{d}{a}$

$1 + (m + r)\lambda + mr\lambda^2 = 1 + 2n\lambda + n^2\lambda^2$

$\Rightarrow \lambda = \frac{m+r-2n}{n^2 - mr}$ ...(1)

But, m, n, r are in A.P.

$\therefore m + r = \frac{2mr}{n}$

$\therefore (1) \Rightarrow \lambda = \frac{\frac{2mr}{n} - 2n}{n^2 - mr} = -\frac{2}{n}$.

79. (c): $1, a_1, a_2, ...a_m, 31$ are in A.P. The common difference d is given by $31 = 1 + (m + 1)d$

$\Rightarrow d = \frac{30}{m+1}$

$\frac{a_7}{a_{m-1}} = \frac{5}{9} \Rightarrow \frac{1+7d}{1+(m-1)d} = \frac{5}{9}$

$\frac{5}{9} = \frac{1 + \frac{210}{m+1}}{1 + \frac{(m-1)30}{m+1}} = \frac{m+211}{31m-29}$

$5(31m - 29) = 9(m + 211)$

$73m = 1022 \Rightarrow m = 14.$

80. (b): $g = \sqrt{ab}$

a, p, q, b are in A.P.

Common difference d is $\frac{b-a}{3}$

$\therefore\ p = a + d = a + \frac{b-a}{3} = \frac{2a+b}{3}$

$q = b - d = b - \frac{b-a}{3} = \frac{a+2b}{3}$

$(2p - q)(p - 2q)$

$= \frac{(4a+2b-a-2b)}{3} \cdot \frac{(2a+b-2a-4b)}{3}$

$= -ab = -g^2.$

81. (a) : $x = \frac{1}{9}(999...9) = \frac{1}{9}(10^{20} - 1)$

$y = \frac{1}{3}(999...9) = \frac{1}{3}(10^{10} - 1)$

$z = \frac{2}{9}(999...9) = \frac{2}{9}(10^{10} - 1)$

$\therefore\ \frac{x - y^2}{z} = \frac{10^{20} - 1 - (10^{10} - 1)^2}{2(10^{10} - 1)}$

$= \frac{10^{10} + 1 - (10^{10} - 1)}{2} = 1.$

82. (a): Let the A.G.P. be 3, $(3 + d)r$, $(3 + 2d)r^2$....

$(3 + d)r = 4$, $(3 + 2d)r^2 = 4$

Eliminating r, $(3 + d)^2 = 4(3 + 2d)$

$\Rightarrow d^2 - 2d - 3 = 0 \Rightarrow d = -1, 3$

$d = -1 \Rightarrow r = 2$. But $|r| < 1$

$\therefore\ d = 3 \Rightarrow r = \frac{2}{3}$

$S = 3 + (3 + d)r + (3 + 2d)r^2 + ...$

$Sr = 3r + (3 + d)r^2 + ...$

Subtracting, $S(1 - r) = 3 + dr + dr^2 + ...$

$= 3 + \frac{dr}{1-r} = 3 + \frac{3 \times \frac{2}{3}}{\left(1 - \frac{2}{3}\right)} = 9$

$S = \frac{9}{1 - \frac{2}{3}} = 27.$

83. (c) : G.P. is $a, ar, ar^2, ...$

$a = ar + ar^2 \Rightarrow r^2 + r - 1 = 0$

$\Rightarrow r = \frac{\sqrt{5}-1}{2}$ since $r > 0$

Assertion & Reason

1. (d) : Let n be the number of sides

$\frac{n}{2}[240° + (n-1)5°] = (n-2)180°$

$\Rightarrow n^2 - 25n + 144 = 0 \quad \therefore n = 9,\ 16$

But $n = 16 \Rightarrow$ the largest angle is $120° + 15 \times 5° = 195° > 180°$ which is not possible in a convex polygon

$\therefore\ n = 9.$

2. (a) : $\sqrt{3} = (1+d)r \quad 2 = (1+2d)r^2$

$\Rightarrow \frac{2\sqrt{3}}{r} = 2 + 2d, \quad \frac{2}{r^2} = 1 + 2d$

$\Rightarrow 1 = \frac{2\sqrt{3}}{r} - \frac{2}{r^2}, \quad r^2 - 2\sqrt{3}r + 2 = 0$

$r = \sqrt{3} \pm 1$

Taking $r = \sqrt{3} - 1$, $d = \frac{\sqrt{3}}{r} - 1$

$\therefore\ d = \frac{\sqrt{3}}{\sqrt{3}-1} - 1 = \frac{\sqrt{3}+1}{2}$

$x = (1+3d)r^3 = \left[1 + \frac{3}{2}(\sqrt{3}+1)\right](\sqrt{3}-1)^3$

$= \left(\frac{5+3\sqrt{3}}{2}\right)(\sqrt{3}-1)(4-2\sqrt{3})$

$= (5+3\sqrt{3})(\sqrt{3}-1)(2-\sqrt{3})$

$= (3\sqrt{3}+5)(3\sqrt{3}-5) = 27 - 25 = 2.$

3. (c) : S_1 is true. It can be shown that S_2 is false.

$\sqrt{2} - \sqrt{3} = (l - m)d$

$\sqrt{3} - \sqrt{5} = (m - n)d$

On division we get, $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{3}-\sqrt{5}} = \frac{l-m}{m-n}$

$\Rightarrow$ Irrational = Rational, which is not possible.

4. (d) : Let $S_n = a_1 + a_2 + a_3 + ... + a_n$

$\Rightarrow \sum n^2 - S_n = \frac{1}{3}n(n^2 - 1)$

$\Rightarrow S_n = \frac{n(n+1)(2n+1)}{6} - \frac{1}{3}n(n+1)(n-1)$

$= \frac{n(n+1)}{6}[2n + 1 - 2(n-1)]$

$S_n = \frac{n(n+1)}{2}$

$S_n = 1 + 2 + 3 + ...n = \sum n$

$\Rightarrow t_n = n \neq n + 1$

$\therefore$ Statement-1 is false and Statement-2 is true.

5. (b) : Let a, ar, ar^2 be the three terms of a G.P. (which forms a triangle).

We have $(a, r > 0)$

$a + ar > ar^2$ (assume $r \geq 1$

(Sum of two sides > third side)

$\Rightarrow 1 + r > r^2$

$\Rightarrow r^2 - r - 1 < 0$

$\Rightarrow \frac{1-\sqrt{5}}{2} < r < \frac{1+\sqrt{5}}{2}$

But $r > 1$ hence $1 < r < \frac{1+\sqrt{5}}{2}$

Also r can be less than 1 so that gives $r > \frac{\sqrt{5}-1}{2}$

6. **(d) :** $\because \frac{S_n}{S'_n} = \frac{3n+8}{7n+15} = \frac{n(3n+8)}{n(7n+15)} = \frac{3n^2+8n}{7n^2+15n}$

$\Rightarrow \quad \frac{t_n}{t'_n} = \frac{S_n - S_{n-1}}{S'_n - S'_{n-1}} = \frac{(3n^2+8n)-3(n-1)^2-8(n-1)}{7n^2+15n-7(n-1)^2-15(n-1)}$

$\Rightarrow \quad \frac{t_n}{t'_n} = \frac{6n+5}{14n+8} \Rightarrow t_n : t'_n = 6n + 5 : 14n + 8$

Now Statement-1 is false & Statement-2 is true.

7. **(d) :** $\because$ A.M. $\geq$ G.M. as a, b, c are distinct real numbers

$\therefore \quad \frac{a^2+b^2}{2} > \sqrt{a^2b^2}$ *i.e.* $a^2 + b^2 > 2ab$ .... (i)

Similarly $b^2 + c^2 > 2bc$ .... (ii)

& $c^2 + a^2 > 2ac$ ....(iii)

Now by adding (i), (ii) (iii) we get

$2(a^2 + b^2 + c^2) > 2(ab + bc + ca)$

$\Rightarrow \quad ab + bc + ca < 1$ Using $a^2 + b^2 + c^2 = 1$

$\therefore$ Statement-1 is false & Statement-2 is true.

8. **(d) :** $\because t_n = S_n - S_{n-1}$

$= \frac{n(n+1)(n+2)}{6} - \frac{(n-1)n(n+1)}{6}$

$t_n = \frac{1}{2}n(n+1)$

$\therefore \quad \frac{1}{t_n} = \frac{2}{n(n+1)} = 2\left[\frac{1}{n} - \frac{1}{n+1}\right]$

$\therefore \quad \sum_{r=1}^{n} \frac{1}{t_r} = 2\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+......+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$

$= 2\left[1-\frac{1}{n+1}\right]$

$\therefore \quad \underset{n\to\infty}{lt} \sum_{r=1}^{n} \frac{1}{t_r} = 2 \underset{n\to\infty}{lt}\left[1-\frac{1}{n+1}\right] = 2 \neq 4$

$\therefore$ Statement-1 is false but Statement-2 is correct.

9. **(b) :** Using A.M. > G.M.

$\frac{1^3+2^3+.....+n^3}{n} > (1^3 \cdot 2^3 \cdot 3^3 \cdot ... n^3)^{1/n}$

$\Rightarrow \quad \frac{\Sigma n^3}{n} > (n!)^{3/n}$

$\Rightarrow \quad \frac{n^2(n+1)^2}{4n} > (n!)^{3/n}$

$\Rightarrow \quad \left[n\left(\frac{(n+1)}{2}\right)^2\right]^n > (n!)^3 \Rightarrow \quad n^n\left(\frac{n+1}{2}\right)^{2n} > (n!)^3$

$\Rightarrow \quad (n!)^3 < n^n\left(\frac{n+1}{2}\right)^{2n}$

Hence Statement-1 & Statement-2 both are true but Statement-2 is not proper explanation of (1)

10. **(a) :** $\because$ Sum of n terms of an A.P. is

$S_n = \frac{n}{2}[2A + (n-1)D]$

Where A = first term, D = common difference. Hence sum of n terms of an A.P. is always in the form of quadratic expression & Statement-2 is true.

Again $S_n = 2n^2 + 3n + 1$

$\therefore \quad t_n = S_n - S_{n-1}$

$= 2[n^2 - (n-1)^2] + 3[n - n + 1]$

$= 2[2n - 1] + 3 = 4n + 1$

$\therefore \quad D = t_{n+1} - t_n = 4.$

11. **(a) :** $2^{1/4}\, 4^{1/8}\, 8^{1/16}\, (16)^{1/32} ... \infty = 2^x$

$\Rightarrow \quad 2^x = 2^{1/4} \cdot 2^{2/8} \cdot 2^{3/16} \cdot 2^{4/32} ... \infty \Rightarrow 2^x = 2^{\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+...\infty}$

$\Rightarrow \quad x = \frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+........\infty$ .....(i)

$\Rightarrow \quad \frac{x}{2} = \frac{1}{8}+\frac{2}{16}+\frac{3}{32}+........\infty$ ....(ii)

$\therefore$ (i) – (ii), we have

$\frac{x}{2} = \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+.......\infty$

$\frac{x}{2} = \frac{1}{4}\left(\frac{1}{1-\frac{1}{2}}\right) \Rightarrow x = 1.$

12. **(d) :** $\because \sqrt{2} = A + (a-1)d, \sqrt{3} = A + (b-1)d,$

$\sqrt{5} = A + (c-1)d$

$\therefore \quad \sqrt{2} - \sqrt{3} = (a-b)d$ ....(i)

And $\sqrt{3} - \sqrt{5} = (b-c)d$ ....(ii)

$= \frac{\sqrt{2}-\sqrt{3}}{\sqrt{3}-\sqrt{5}} = \frac{a-b}{b-c}$

$\therefore$ Statement-2 is true but Statement-1 is false.

13. **(d) :** Statement-1 is false as each term of the series $\frac{1}{10^9}+\frac{2}{10^9}+\frac{3}{10^9}+......$ smaller than $\frac{1}{10^8}$ but its sum to infinity terms is infinity and on the other side $\lim_{n\to\infty} \frac{n}{10^8}$ does not exist so statement (2) is true.

QUESTIONS FROM PREVIOUS YEARS AIEEE/JEE MAIN

1. **(a) :** $(1^3 + 3^3 + 5^3 + ... + 9^3) - (2^3 + 4^3 + 6^3 + 8^3)$

$= (1^3 + 3^3 + 5^3 + ... + 9^3) - 2^3(1^3 + 2^3 + 3^3 + 4^3)$

$= [1^3 + 3^3 + ... + (2n-1)^3]_{n = \text{odd} = 5}$

$\qquad - 2^3[1^3 + 2^3 + ... + n^3]_{n = \text{even} = 4}$

$= [2n(n+1)(n+2)(n+3) - 12n(n+1)(n+2)$

$+ 13n(n+1) - n]_{n = 5\,(\text{odd})} - 2^3\left[\frac{n^2(n+1)^2}{4}\right]_{\substack{n=4\\(\text{even})}}$

(Remember this result)

$= [2 \times 5 \times 6 \times 7 \times 8 - 12 \times 5 \times 6 \times 7 + 13 \times 5 \times 6 - 5] - 2^3\left(\frac{16\times 25}{4}\right)$

$= [3750 - 5(505)] - 2 \times 16 \times 25$

$= 1225 - 800 = 425.$

2. **(b) :** Let terms of G.P are $a, ar, ar^2, ...$

$\therefore\ S_\infty = \frac{a}{1-r}$ a = first term, r = common ratio, $S_\infty = 20$

According to question $\frac{a}{1-r} = 20$

$\Rightarrow\ a = 20(1 - r)$ ...(i)

also $\frac{a^2}{1-r^2} = 100$

$\Rightarrow\ \frac{a}{1-r}\cdot\frac{a}{1+r} = 100$

$\Rightarrow\ a = 5(1 + r)$ ...(ii)

Solving (i) and (ii) we have $r = 3/5$

3. **(b):** $S_\infty = 2^{\frac{1}{4}}\cdot 2^{\frac{2}{8}}\cdot 2^{\frac{3}{16}}\cdot 2^{\frac{4}{32}}\cdot ....\cdot\infty = 2^\lambda$ (say) ...(*)

Where $\lambda = \frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32} + ...\infty$ ...(i)

$\frac{\lambda}{2} = 0 + \frac{1}{8}+\frac{2}{16}+\frac{3}{32}+\frac{4}{64} + ...\infty$ ...(ii)

Now (i) – (ii) $\Rightarrow \frac{\lambda}{2} = \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+...$

$\frac{\lambda}{2} = \frac{a}{1-r} = \frac{1}{4}\times\frac{2}{1}$ $\therefore$ $\lambda = 1$ so $S_\infty = 2^1$

4. **(b) :** Let first term of an G.P is a and common ratio r $\therefore$ $t_5 = ar^4 = 2$

$\therefore\ \prod_{i=1}^{9} a_i = a\cdot ar\ ar^2\ .....ar^8$

$= a^9 r^{\frac{8\times 9}{2}} = a^9 r^{36} = (ar^4)^9 = 2^9$

5. **(d) :** Let the polynomial be $f(x) = ax^2 + bx + c$

Given $f(1) = f(-1) \Rightarrow b = 0$

$\therefore\ f(x) = ax^2 + c$

Now $f'(x) = 2ax$

$\therefore\ f'(a) = 2a^2, f'(b) = 2ab, f'(c) = 2ac$

as $a, b, c \in$ A.P.

$\Rightarrow\ a^2, ab, ac \in$ A.P.

$\Rightarrow\ 2a^2, 2ab, 2ac \in$ A.P.

$\Rightarrow\ f'(a), f'(b), f'(c) \in$ A.P.

6. (c) : **I[st] Solution :** $S = \frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4} - ...\infty$

Let $S_1 = \frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6} + ...\infty$

(sum of positive terms)

$\therefore\ T_n = \frac{1}{(2n-1)(2n)} = \frac{1}{2n-1}-\frac{1}{2n}$

$\therefore\ S_1 = \Sigma T_n = \sum\left(\frac{1}{2n-1}-\frac{1}{2n}\right)$

$= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\ .....\infty$

$= \log_e 2$ ...(i)

Again $S_2 = \frac{1}{2.3}+\frac{1}{4.5}+\frac{1}{6.7} + ...\infty$

(sum of negative terms)

$T'_n = \frac{1}{(2n)(2n+1)}$

$S_2 = \Sigma T_n' = \sum\frac{1}{(2n)(2n+1)} = \sum\left(\frac{1}{2n}-\frac{1}{2n+1}\right)$

$= \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+....\infty$

$= -\left[-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+...\infty\right]$

$= -[\log_e 2 - 1] = 1 - \log_e 2$ ...(ii)

Now $S = S_1 - S_2$ (i) – (ii)

$= \log_e 2 - 1 + \log_e 2 = \log_e(4/e)$

II[nd] Solution :

Consider $\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{3.4}-\frac{1}{4.5} + ...\infty$

$= \frac{2-1}{1.2}-\frac{3-2}{2.3}+\frac{4-3}{3.4}-\frac{5-4}{4.5} + ...\infty$

$= \left(1-\frac{1}{2}\right)-\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right) + ...\infty$

$= 1-2\left[\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+....\infty\right]$

$= 1 + 2\log_e 2 - 2 = 2\log_e 2 - 1 = \log_e(4/e)$

III[rd] Solution :

$T_n = \frac{(-1)^{n+1}}{n(n+1)}$ $\therefore$ $S_n = \sum\frac{(-1)^{n+1}}{n(n+1)}$

$\therefore\ S_n = \sum\frac{(-1)^{n+1}}{n}+\sum\frac{(-1)^n}{n+1}$

$= \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...\infty\right)+\left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...\infty\right)$

$= 2\log_e 2 - 1 = \log_e(4/e).$

7. **(a) :** According to problem $f(x) = x + 1/x$

$\therefore\ \frac{x+\frac{1}{x}}{2} \geq x\cdot 1/x$ (A.M. $\geq$ G.M.)

$\Rightarrow\ x + \frac{1}{x} \geq 2$

$\Rightarrow\ x = 1$ for $f(x)$ to be minimum.

8. (c): $\frac{a_2}{a_1}=\frac{a_3}{a_2}=\dots-\frac{a_n}{a_{n-1}} = r$

which means $a_n, a_{n+1}, a_{n+2} \in$ G.P.

$\Rightarrow a_{n+1}^2 = a_n a_{n+2}$

$\Rightarrow 2\log a_{n+1} - \log a_n - \log a_{n+2} = 0$ ...(i)

Similarly

$2\log a_{n+4} - \log a_{n+3} - \log a_{n+5} = 0$ ....(ii)

and

$2\log a_{n+7} - \log a_{n+6} - \log a_{n+8} = 0$ ...(iii)

Using $C_1 \to C_1 + C_3 - 2C_2$, we get $\Delta = 0$

9. (c) : $T_m = a + (m-1)d = \frac{1}{n}$ ...(i)

$T_n = a + (n-1)d = \frac{1}{m}$ ...(ii)

Now $T_m - T_n = \frac{1}{n} - \frac{1}{m} = (m-n)\,d$

$\Rightarrow d = \frac{1}{mn}$ and $a = \frac{1}{mn}$

$\therefore\ a - d = 0$

10. (b) : As S_n is needed when n is odd let $n = 2k + 1$

$\therefore\ S_n = S_{2k+1} =$ Sum up to $2k$ terms + $(2k+1)^{\text{th}}$ term

$= \frac{2k(2k+1)^2}{2}$ + last term

$= \frac{(n-1)n^2}{2} + n^2$ as $n = 2k + 1$

$= \frac{n^2(n+1)}{2}$

Alternative Solution :

We have $T_1 = 1^2$, $T_3 = 3^2$, $T_5 = 5^2$

let n is odd then $(n-1)$ will be even and

$S_{n-1} = \frac{(n-1)(n-1+1)^2}{2} = \frac{(n-1)n^2}{2}$

$\therefore\ S_n = S_{n-1} + t_n$

$= \frac{(n-1)n^2}{2} + n^2 = \frac{(n+1)n^2}{2}$

11. (a) : Given $|a| < 1$, $|b| < 1$, $|c| < 1$, $a, b, c \in$ A.P.

and $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$, $\sum_{n=0}^{\infty} b^n = \frac{1}{1-b}$, $\sum_{r=0}^{\infty} c^n = \frac{1}{1-c}$

$\therefore\ x = \frac{1}{1-a},\ y = \frac{1}{1-b},\ z = \frac{1}{1-c}$

$\Rightarrow a = \frac{x-1}{x},\ b = \frac{y-1}{y},\ c = \frac{z-1}{z}$

$\therefore\ 2b = a + c$ as $a, b, c \in$ A.P.

$2\left(\frac{y-1}{y}\right) = \frac{x-1}{x} + \frac{z-1}{z} \Rightarrow \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$

$\Rightarrow x, y, z \in$ H.P.

Alternative Solution :

Given $x = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

Similarly $y = \frac{1}{1-b}$ and $z = \frac{1}{1-c}$

Now $a, b, c \in$ A.P. $\Rightarrow -a, -b, -c \in$ A.P.

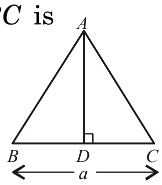
$\Rightarrow 1-a, 1-b, 1-c \in$ A.P.

$\Rightarrow \frac{1}{1-a}, \frac{1}{1-b}, \frac{1}{1-c} \in$ H.P.

(If $a, b, c \in$ A.P. then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c} \in$ H.P.)

$\Rightarrow x, y, z \in$ H.P.

12. (c) : Altitude from A to BC is



Area of $\Delta = \frac{1}{2} AD \times BC$

$\therefore\ \frac{2 \cdot \text{Area of } \Delta}{a} = AD$

$\therefore$ Altitudes are in H.P.

$\therefore\ \frac{2\Delta}{a}, \frac{2\Delta}{b}, \frac{2\Delta}{c} \in$ H.P. $\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \in$ H.P.

$\Rightarrow a, b, c \in$ A.P.

13. (d) : Given $a_1, a_2, a_3, \dots$ be terms of A.P.

$\frac{a_1 + a_2 + \dots a_p}{a_1 + a_2 + \dots a_q} = \frac{p^2}{q^2}$

$\Rightarrow \frac{\frac{p}{2}[2a_1 + (p-1)d]}{\frac{q}{2}[2a_1 + (q-1)d]} = \frac{p^2}{q^2} \Rightarrow \frac{2a_1 + (p-1)d}{2a_1 + (q-1)d} = \frac{p}{q}$

$\Rightarrow [2a_1 + (p-1)d]q = p[2a_1 + (q-1)d]$

$\Rightarrow 2a_1(q-p) = d[(q-1)p - (p-1)q]$

$\Rightarrow 2a_1(q-p) = d(q-p) \Rightarrow 2a_1 = d$

$\Rightarrow \frac{a_6}{a_{21}} = \frac{a_1 + 5d}{a_1 + 20d} = \frac{a_1 + 10a_1}{a_1 + 40a_1} \Rightarrow \frac{a_6}{a_{21}} = \frac{11}{41}.$

14. (d) : Given $a_1, a_2, \dots, a_n$ are in H.P.

$\Rightarrow \frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n} \in$ A.P.

$\Rightarrow \frac{1}{a_2} - \frac{1}{a_1} = d \Rightarrow a_1 a_2 = \frac{a_1 - a_2}{d} = \frac{a_1}{d} - \frac{a_2}{d}$ ...(i)

$a_2 a_3 = \frac{a_2}{d} - \frac{a_3}{d}$ ...(ii)

$\vdots$

$a_{n-1} a_n = \frac{a_{n-1}}{d} - \frac{a_n}{d}$ ...(n)

Adding (i), (ii), ..., (n) equations we get

$a_1a_2 + a_2a_3 + a_3a_4 + \dots a_{n-1}a_n = \frac{a_1}{d} - \frac{a_n}{d}$

Also $\frac{1}{a_n} = \frac{1}{a_1} + (n-1)d \Rightarrow \frac{a_1 - a_n}{d} = (n-1)a_1a_n$

$\therefore\ a_1a_2 + a_2a_3 + \dots a_{n-1}a_n = (n-1)a_1a_n.$

15. (b) : Given, $a = ar + ar^2 \Rightarrow r^2 + r - 1 = 0$

$\Rightarrow r = \frac{-1+\sqrt{5}}{2}.$

16. (b) : Let the first term of the G.P. be 'a' and the common ratio be 'r'

$\therefore$ We have

$a + ar = 12$ ...(1)

$ar^2 + ar^3 = 48$ ...(2)

Solving the equations (1) and (2) we get $r = \pm 2$

But, as the terms are alternative positive and negative

$\therefore$ r is negative $\Rightarrow$ $r = -2$

Put $r = -2$ in equation (1)

$a(1 - 2) = 12 \Rightarrow a = -12$

17. (a) : Let $S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + ...$

$\frac{1}{3}S = \quad \frac{1}{3} + \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + ...$

$-$

$\frac{2}{3}S = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + ...$

$= \frac{4}{3}\left[1 + \frac{1}{3} + ... \text{to } \infty\right]$

$= \frac{4}{3} \cdot \frac{1}{1 - \frac{1}{3}} = \frac{4}{3} \cdot \frac{1}{2/3} = 2$

$\Rightarrow \frac{2}{3}S = 2 \quad \therefore \quad S = 3.$

18. (b) : We have $a_1 + a_2 + ... + a_n = 4500$

$\Rightarrow a_{11} + a_{12} + ... + a_n = 4500 - 10 \times 150 = 3000$

$\Rightarrow 148 + 146 + ... = 3000$

$\Rightarrow \frac{n-10}{2} \cdot (2 \times 148 + (n - 10 - 1)(-2)) = 3000$

Let $n - 10 = m$

$\Rightarrow m \times 148 - m(m - 1) = 3000$

$\Rightarrow m^2 - 149m + 3000 = 0$

$\Rightarrow (m - 24)(m - 125) = 0$

$\therefore m = 24, 125,$

We get, $n = 34, 135$

But for $n = 135$, we have

$a_{135} = 148 + (135 - 1)(-2) = 148 - 268 < 0$

But a_{34} is positive.

Hence, $n = 34$ is the only answer.

19. (b) : Let it happen after n months.

$3 \times 200 + \frac{n-3}{2}\{2 \times 240 + (n - 4)40\} = 11040$

$\Rightarrow \left(\frac{n-3}{2}\right)(480 + 40n - 160) = 11040 - 600$

$= 10440$

$\Rightarrow n^2 + 5n - 546 = 0 \Rightarrow (n + 26)(n - 21) = 0$

$\therefore n = 21.$

20. (d) : Statement 1 :

$1 + (1 + 2 + 4) + (4 + 6 + 9) + ... + (361 + 380 + 400)$ is 8000

Statement 2 : $\sum_{k=1}^{n} (k^3 - (k-1)^3) = n^3$

Statement 1 : $T_1 = 1$, $T_2 = 7 = 8 - 1$, $T_3 = 19 = 27 - 8 \Rightarrow T_n = n^3 - (n - 1)^3$

$\therefore$ Statement 2 is a correct explanation of statement 1.

21. (b) : $100\,(a + 99d) = 50\,(a + 49d)$

$\Rightarrow a + 149d = 0$ *i.e.*, $T_{150} = 0$

22. (b) : $t_r = \underbrace{0.77777....7}_{r \text{ terms}}$

$= \frac{7}{10} + \frac{7}{10^2} + ... + \frac{7}{10^r} = \frac{7}{9}(1 - 10^{-r})$

$S_{20} = \sum_{r=1}^{20} t_r = \frac{7}{9}\left(20 - \sum_{r=1}^{20} 10^{-r}\right) = \frac{7}{9}\left\{20 - \frac{1}{9}(1 - 10^{-20})\right\}$

$= \frac{7}{81}(179 + 10^{-20})$

23. (c) : Let the number be a, ar, ar^2, we have

$2|2ar| = a + ar^2 \Rightarrow 4r = r^2 + 1$

$\Rightarrow r^2 - 4r + 1 = 0$

$\Rightarrow r = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

$\therefore r = 2 + \sqrt{3}$, as number is positive.

24. (b) : Let $P = 10^9 + 2 \cdot 11 \cdot 10^8 + ... + 10 \cdot 11^9$

we have $\frac{11}{10}P = 11 \cdot 10^8 + ... + 9 \cdot 11^9 + 11^{10}$

On subtracting, we get

$\Rightarrow \frac{1}{10}P = 11^{10} - [10^9 + 11^1 \cdot 10^8 + 11^2 \cdot 10^7 + ... + 11^9]$

$= 11^{10} - 10^9 \left\{\frac{\left(\frac{11}{10}\right)^{10} - 1}{\frac{11}{10} - 1}\right\} = 11^{10} - 11^{10} + 10^{10}$

$\therefore P = 10^{11} = (100) \cdot 10^9$

On comparison, $k = 100$

25. (d) : The n^{th} term, t_n is

$= \frac{1^3 + 2^3 + ... + n^3}{1 + 3 + ... + (2n - 1)} = \frac{\frac{n^2(n+1)^2}{4}}{n^2} = \frac{(n+1)^2}{4}$

$\sum_{n=1}^{9} t_n = \sum_{n=1}^{9} \frac{(n+1)^2}{4} = \frac{1}{4}\left\{\sum_{n=1}^{10} n^2 - 1\right\}$

$= \frac{1}{4}\left\{\frac{10 \cdot 11 \cdot 21}{6} - 1\right\} = \frac{1}{4}\{385 - 1\} = \frac{1}{4} \times 384 = 96$

26. (d) : Given that l, G_1, G_2, G_3, n are in G.P.

$G_1 = lr$, $G_2 = lr^2$, $G_3 = lr^3$, $n = lr^4$

Then $G_1^4 + 2G_2^4 + G_3^4 = (lr)^4 + 2(lr^2)^4 + (lr^3)^4$

$= (l^3)(lr^4) + 2l^2(lr^4)^2 + l \cdot (lr^4)^3$

$= l^3 \cdot n + 2l^2 \cdot n^2 + ln^3$

$= ln(l^2 + 2nl + n^2) = ln(n + l)^2 = 4m^2nl$

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